This exam has 4 problems. Show all your work and simplify your answers. Answers with missing or insufficient justification will receive no points. You are allowed one 8.5×11-in page of notes (ONE side). You may NOT use a calculator, smartphone, smartwatch, the Internet or any other electronic device.

Problem 1 (30 pts)
Consider the function

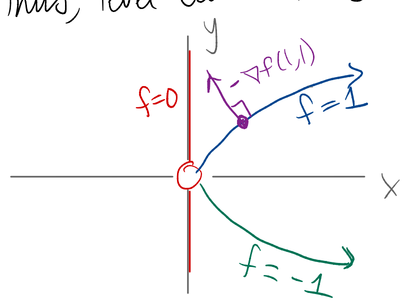
$$f(x, y) = \frac{\sqrt{x}}{y}$$

(a) Graph the level curve of $f(x, y)$ that passes through the point $(0, 2)$. Label the value of f along the curve.
(b) On the same graph as part (a) graph the level curve where $f(x, y) = 1$. Label the value of f along this curve.
(c) On the same graph as part (a), graph one level curve where $f(x, y) < 0$. Label the value of f along this curve.
(d) At the point $(1, 1)$, give a vector that points in the direction in the domain where this function *decreases* fastest.
(e) Sketch the vector you found in part (d) starting at $(1, 1)$ on your graph from part (a).
(f) Use a *2nd order (i.e. quadratic)* Taylor approximation centered at $(1, 1)$ to approximate $\frac{\sqrt{1.8}}{1.5}$
You can leave your answer as an unsimplified sum and/or difference of terms.

1a).

$f(0,2) = \frac{\sqrt{0}}{2} = 0$

Thus, level curve is $0 = \frac{\sqrt{x}}{y} \Rightarrow \sqrt{x} = 0 \Rightarrow x = 0$



Domain of $f(x,y) = \frac{\sqrt{x}}{y}$ is $y \neq 0$, $x \geq 0$
so not defined at origin
or along entire x-axis

b). $1 = \frac{\sqrt{x}}{y} \Rightarrow y = \sqrt{x}$

c). $-1 = \frac{\sqrt{x}}{y} \Rightarrow y = -\sqrt{x}$

d). $-\nabla f(1,1)$ will point in direction of greatest decrease

$\nabla f = \langle \frac{1}{2\sqrt{x}\,y}, \frac{-\sqrt{x}}{y^2} \rangle$

$\Rightarrow \nabla f(1,1) = \langle \frac{1}{2}, -1 \rangle$

$\Rightarrow \boxed{-\nabla f(1,1) = \langle -\frac{1}{2}, 1 \rangle}$

e). See sketch above

(f) Use a *2nd order (i.e. quadratic)* Taylor approximation centered at $(1,1)$ to approximate $\frac{\sqrt{1.8}}{1.5}$.
You can leave your answer as an unsimplified sum and/or difference of terms.

f). $Q(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1) + \frac{1}{2}f_{xx}(1,1)(x-1)^2 + f_{xy}(1,1)(x-1)(y-1) + \frac{1}{2}f_{yy}(1,1)(y-1)^2$

$\left.\begin{array}{l} f(1,1) = 1 \\ f_x(1,1) = \frac{1}{2} \\ f_y(1,1) = -1 \end{array}\right\}$ from part d

$f_{xx} = -\frac{1}{4x^{3/2}y} \Rightarrow f_{xx}(1,1) = -\frac{1}{4}$

$f_{xy} = -\frac{1}{2\sqrt{x}\,y^2} \Rightarrow f_{xy}(1,1) = -\frac{1}{2}$

$f_{yy} = \frac{2\sqrt{x}}{y^3} \Rightarrow f_{yy}(1,1) = 2$

$\Rightarrow Q(x,y) = 1 + \frac{1}{2}(x-1) - 1(y-1) - \frac{1}{8}(x-1)^2 - \frac{1}{2}(x-1)(y-1) + 1(y-1)^2$

$f(1.8, 1.5) \approx Q(1.8, 1.5)$

$= 1 + \frac{1}{2}(0.8) - 1(0.5) - \frac{1}{8}(0.8)^2 - \frac{1}{2}(0.8)(0.5) + (0.5)^2$

$\boxed{= 0.87}$

Problem 2 (22 pts) The temperature (in degrees Farenheit) in a region in space is given by

$$T(x,y,z) = \frac{1}{2}x^2 + \frac{1}{2}xyz$$

A particle is moving in this region and its position at time t is given by

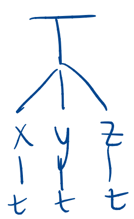
$$\overrightarrow{\mathbf{r}}(t) = 2\cos(\pi t)\mathbf{i} + e^{(9-t^2)}\mathbf{j} - 2t\mathbf{k}$$

where time is measured in seconds and distance in meters.

(a) Use the chain rule to determine how fast the temperature experienced by the particle is changing in degrees Farenheit *per second* at the point $(x,y,z) = (-2,1,-6)$.

(b) How fast is the temperature experienced by the particle changing in degrees Farenheit *per meter* at the point $(x,y,z) = (-2,1,-6)$? (i.e. find the rate of change of the temperature with respect to distance in the direction the particle is moving at the point $(x,y,z) = (-2,1,-6)$).

a). Want $\frac{dT}{dt}$ where $T(x,y,z) = \frac{1}{2}x^2 + \frac{1}{2}xyz$ and $x(t) = 2\cos(\pi t)$, $y(t) = e^{9-t^2}$, $z(t) = -2t$



when $(x,y,z) = (-2,1,-6) \Rightarrow \left.\begin{aligned} -2 &= 2\cos(\pi t) \\ 1 &= e^{9-t^2} \\ -6 &= -2t \end{aligned}\right\} \Rightarrow t = 3$

$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt}$$

$$\Rightarrow \frac{dT}{dt} = \left(x + \tfrac{1}{2}yz\right)\left(-2\pi\sin(\pi t)\right) + \left(\tfrac{1}{2}xz\right)\left(-2te^{9-t^2}\right) + \left(\tfrac{1}{2}xy\right)(-2)$$

$$\Rightarrow \left.\frac{dT}{dt}\right|_{\substack{t=3 \\ (x,y,z)=(-2,1,-6)}} = (-5)(0) + (6)(-6) + (-1)(-2) = \boxed{-34\ {}^\circ F/s}$$

b). Want $D_{\hat{u}}T$ in direction particle is moving

Particle is moving in direction $\vec{r}\,'(t) = \langle -2\pi\sin(\pi t), -2te^{9-t^2}, -2 \rangle$

$(x,y,z) = (-2,1,-6) \Rightarrow t = 3$

Thus, let $\hat{u} = \dfrac{\vec{r}\,'(3)}{\|\vec{r}\,'(3)\|} = \dfrac{\langle 0,-6,-2\rangle}{\sqrt{(-6)^2+(-2)^2}} = \dfrac{\langle 0,-6,-2\rangle}{2\sqrt{10}}$

$$D_{\hat{u}}T(-2,1,-6) = \nabla T(-2,1,-6)\cdot\hat{u} = \langle -5,6,-1\rangle \cdot \frac{\langle 0,-6,-2\rangle}{2\sqrt{10}} = \frac{-34}{2\sqrt{10}} = \frac{-17}{\sqrt{10}} = \boxed{\frac{-17\sqrt{10}}{10}\ \frac{{}^\circ F}{m}}$$

Problem 3 (28 pts)
The following parts are not related:

(a) Find and classify all critical points of

$$g(x, y) = x^4 + y^4 - 4xy$$

$\nabla g = \langle 4x^3 - 4y,\ 4y^3 - 4x \rangle$

$\nabla g = \vec{0} \Rightarrow \boxed{\begin{array}{l} 4x^3 - 4y = 0 \\ 4y^3 - 4x = 0 \end{array}} \Rightarrow y = x^3$

substitute

$\Rightarrow 4(x^3)^3 - 4x = 0$

$\Rightarrow 4x(x^8 - 1) = 0$

$\Rightarrow x = 0$ or $x^8 = 1$

$\Rightarrow x = \pm 1$

critical pts

since $y = x^3$

$x = 0 \Rightarrow y = 0 \Rightarrow (0,0)$

$x = 1 \Rightarrow y = 1 \Rightarrow (1,1)$

$x = -1 \Rightarrow y = -1 \Rightarrow (-1,-1)$

Classify:

$g_{xx} = 12x^2$

$g_{xy} = -4$

$g_{yy} = 12y^2$

$D = (g_{xx})(g_{yy}) - (g_{xy})^2$

$D = (12x^2)(12y^2) - (-4)^2$

$= 144x^2y^2 - 16$

$(x,y) = (0,0)$

$\Rightarrow D = -16 < 0$

$\Rightarrow$ $\boxed{(x,y) = (0,0) \text{ is a saddle point}}$

$(x,y) = (1,1)$

$D = 144 - 16 > 0$

and $g_{xx}(1,1) = 12 > 0$

$\Rightarrow$ $\boxed{(x,y) = (1,1) \text{ is a local min}}$

$(x,y) = (-1,-1)$

$D = 144 - 16 > 0$

and $g_{xx}(-1,-1) = 12 > 0$

$\Rightarrow$ $\boxed{(x,y) = (-1,-1) \text{ is a local min}}$

(b) An airplane moves in a trajectory given by

$$\vec{\mathbf{r}}(t) = 4t\mathbf{i} + t\mathbf{j} + t^2\mathbf{k} \quad t \geq 0$$

Given this trajectory, it will intersect the following surface twice:

$$z = 2x + 2y - y^2 - 8$$

Determine the tangent plane to the surface at the location where the airplane intersects the surface for a second time. Give your answer in standard (i.e. linear) form.

3b). $r(t) = \langle 4t, t, t^2 \rangle \Rightarrow x(t) = 4t,\ y(t) = t,\ z(t) = t^2$

$$z = 2x + 2y - y^2 - 8$$

Intersection occurs when:

$$t^2 = 2(4t) + 2t - t^2 - 8$$

$$\Rightarrow 2t^2 - 10t + 8 = 0$$

$$\Rightarrow 2(t^2 - 5t + 4) = 0$$

$$\Rightarrow 2(t-4)(t-1) = 0 \quad \Rightarrow t = 1 \text{ or } t = 4$$

2nd intersection occurs at $t = 4$

To find tangent plane, need point on plane & normal vector

point on plane: $\vec{r}(4) = \langle 16, 4, 16 \rangle$

$\Rightarrow (x, y, z) = (16, 4, 16)$ is a point on the plane

normal vector:

$$z = 2x + 2y - y^2 - 8 \quad \Rightarrow \underbrace{2x + 2y - y^2 - 8 - z}_{g(x,y,z)} = 0$$

thus the given surface is one level surface of $g(x,y,z)$.

∇g is normal to the level surfaces of g

$\nabla g = \langle 2, 2-2y, -1 \rangle \Rightarrow \nabla g(16,4,16) = \langle 2, -6, -1 \rangle$ normal to level surface $g = 0$

Thus tangent plane is

$$2(x-16) - 6(y-4) - 1(z-16) = 0$$

$$\Rightarrow \boxed{2x - 6y - z = -8}$$

Problem 4 (20 pts)

A mother puts her child on an amusement park ride that takes the child along a path in the xy-plane described by the equation $x^2 - 2x = 4y - y^2$. While the child is on the ride, the mother stands at the location $(x, y) = (0, 0)$.

(a) Use Lagrange multipliers to find the minimum and maximum distances from the mother to the child during the ride.

(b) Give the (x, y) coordinates of the child at the minimum and maximum distances.

a). $x^2 - 2x = 4y - y^2 \Rightarrow \underbrace{x^2 + y^2 - 2x - 4y}_{g(x,y)} = 0$

Want to optimize $\underset{\text{distance to origin}}{d(x,y) = \sqrt{(x-0)^2 + (y-0)^2}}$ subject to constraint $g(x,y) = 0$

Easier to optimize $[d(x,y)]^2$ (+ will result in same (x,y) locations for max/min)

Let $f(x,y) = [d(x,y)]^2 = x^2 + y^2$

Solve

$\nabla f = \lambda \nabla g$
$g = 0$

$\Rightarrow \boxed{\begin{aligned} 2x &= \lambda(2x-2) \\ 2y &= \lambda(2y-4) \\ x^2 + y^2 - 2x - 4y &= 0 \end{aligned}}$

$\Rightarrow \lambda = \frac{2x}{2x-2} = \frac{x}{x-1}, \ x \neq 1$

$\Rightarrow \lambda = \frac{y}{y-2}, \ y \neq 2$

notice $x = 1$ is not a sol'n of the first eqn
and $y = 2$ " " second eqn
thus we know $x \neq 1, y \neq 2$

$\lambda = \frac{x}{x-1} = \frac{y}{y-2}$

$\Rightarrow \cancel{xy} - 2x = \cancel{xy} - y$

$\Rightarrow \boxed{y = 2x}$

Substitute into constraint:

$x^2 + y^2 - 2x - 4y = 0$

$\Rightarrow x^2 + (2x)^2 - 2x - 4(2x) = 0$

$\Rightarrow 5x^2 - 10x = 0$

$\Rightarrow 5x(x-2) = 0$

$\Rightarrow x = 0$ or $x = 2$

$x = 0 \Rightarrow y = 0 \Rightarrow \boxed{(x,y) = (0,0)}$

$x = 2 \Rightarrow y = 4 \Rightarrow \boxed{(x,y) = (2,4)}$

(x,y)	$d = \sqrt{x^2+y^2}$
$(0,0)$	0
$(2,4)$	$\sqrt{20} = 2\sqrt{5}$

a). Min distance is 0
Max distance is $2\sqrt{5}$

b). $(x,y) = (0,0)$ at min
$(x,y) = (2,4)$ at max