APPM 2350—Exam 1

Friday, June 10th 1-2:35pm 2022

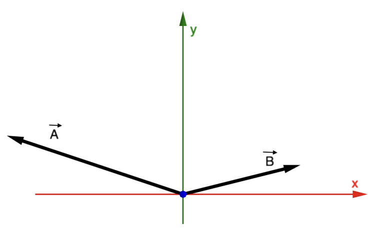
This exam has 4 problems. Show all your work and simplify your answers. Answers with no justification will receive no points. You are allowed one 8.5×11-in page of notes (ONE side). You may NOT use a calculator, smartphone, smartwatch, the Internet or any other electronic device.

Question 1 (20 pts)

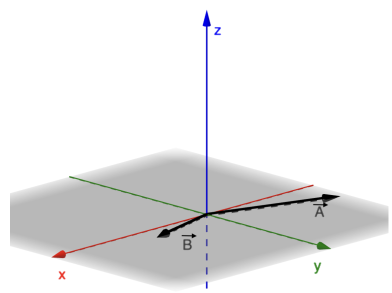
The following parts are not related:

(a) The vectors $\overrightarrow{\mathbf{A}}$ and $\overrightarrow{\mathbf{B}}$, shown below, are parallel to the xy-plane:

2D view



3D view of the same vectors



On your own sheet of paper, sketch and clearly label diagrams for the following (do not draw on this exam sheet). Clearly label each vector and each axis.

(i) $\overrightarrow{\mathbf{A}}$, $\overrightarrow{\mathbf{B}}$, and $\overrightarrow{\mathbf{B}} - \overrightarrow{\mathbf{A}}$

(ii) $\overrightarrow{\mathbf{A}}$, $\overrightarrow{\mathbf{B}}$, and $\mathbf{proj}_{\overrightarrow{\mathbf{B}}}\overrightarrow{\mathbf{A}}$.

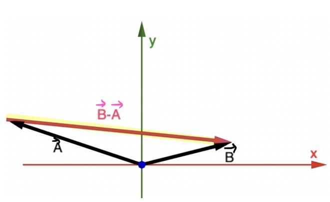
(iii) $\overrightarrow{\mathbf{A}}$, $\overrightarrow{\mathbf{B}}$, and $\overrightarrow{\mathbf{A}} \times \overrightarrow{\mathbf{B}}$

(b) Find one possible force vector $\overrightarrow{\mathbf{F}}$ that satisfies both of the following criteria:

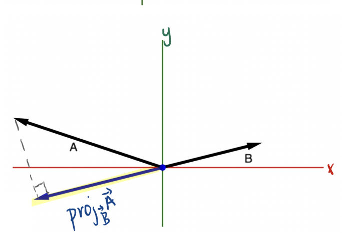
- The work done by $\overrightarrow{\mathbf{F}}$ in moving an object from the point $(2, -2, 4)$ to the point $(-2, 5, 6)$ is 12 N-m (where distances are measured in meters and force is measured in Newtons)
- $\overrightarrow{\mathbf{F}}$ is not parallel to the object's displacement

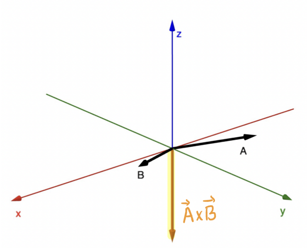
SOLUTION:

(a) (i)



(ii)



(iii) 

(b) The object's displacement is given by

$$\overrightarrow{\mathbf{D}} = \langle -2-2, 5+2, 6-4 \rangle = \langle -4, 7, 2 \rangle$$

Let

$$\overrightarrow{\mathbf{F}} = \langle a, b, c \rangle$$

$$Work = \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{D}}$$

Thus using the algebraic definition of the dot product we want to find any a, b, c such that

$$\overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{D}} = -4a + 7b + 2c = 12 \quad \text{and } \overrightarrow{\mathbf{F}} \text{ is not parallel to } \overrightarrow{\mathbf{D}}$$

$\boxed{\text{There are an infinite number of possible solutions. One possibility: } \overrightarrow{\mathbf{F}} = \langle 0, 0, 6 \rangle}$

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Question 2 (32 pts)
A drone travels along the path given by

$$\overrightarrow{\mathbf{r}}(t) = (t+1)\hat{\mathbf{i}} + 2t\hat{\mathbf{j}} + (2 + 3t - t^2)\hat{\mathbf{k}}$$

(a) Are there any time(s), t, such that the drone's velocity is orthogonal to its acceleration? If so, find these time(s). If not explain why.
(b) The drone's entire trajectory actually lies in one plane. Find the equation of that plane. (Hint: You do not need to find $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$ to do this problem).
(c) At time $t = 2$ the drone fires a missile straight ahead along a line.
  (i) How far from the point $(1, 1, 1)$ is the drone at the moment it fires the missile?
  (ii) Give a vector-valued function that traces out the missile's path.
  (iii) At what point(s) (x, y, z) does the missile intersect the surface $z + 5 = (y-9)^2 - (x-5)^2$?

__SOLUTION:__

(a) We want to find all time(s) such that $\overrightarrow{\mathbf{v}}(t) \cdot \overrightarrow{\mathbf{a}}(t) = 0$

$$\overrightarrow{\mathbf{v}}(t) = \overrightarrow{\mathbf{r}}'(t) = \langle 1, 2, 3-2t \rangle$$

$$\overrightarrow{\mathbf{a}}(t) = \overrightarrow{\mathbf{r}}''(t) = \langle 0, 0, -2 \rangle$$

$$\implies \overrightarrow{\mathbf{v}}(t) \cdot \overrightarrow{\mathbf{a}}(t) = -2(3-2t)$$

$$-2(3-2t) = 0 \implies \boxed{t = \frac{3}{2}}$$

(b) To find the equation of a plane we need a point and a normal vector.

POINT: We can evaluate the position function at any value of t. For example, $\vec{\mathbf{r}}(0) = \langle 1, 0, 2\rangle \implies (x, y, z) = (1, 0, 2)$ is a point on the plane.

NORMAL VECTOR: To find the normal vector, we need to find 2 vectors in the plane and take their cross product. There are multiple ways to find 2 vectors that lie on the plane:

- OPTION 1: Find 3 points on the curve and find 2 vectors between these points.
- OPTION 2: Evaluate $\vec{\mathbf{v}}(t)$ at two different times
- OPTION 3: Since the entire trajectory remains in one plane, the velocity and the acceleration vectors both lie in that plane, so we can take the cross product of those 2 vectors. We will show this option below:

$$\vec{\mathbf{n}} = \vec{\mathbf{v}}(t)\times\vec{\mathbf{a}}(t) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2 & 3-2t \\ 0 & 0 & -2 \end{vmatrix} = [(2)(-2)-(0)(3-2t)]\,\hat{\mathbf{i}} - [(1)(-2)-(0)(3-2t)]\,\hat{\mathbf{j}} + [(1)(0)-(0)(2)]\,\hat{\mathbf{k}}$$

$$= -4\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 0\hat{\mathbf{k}}$$

Using the point we found above we get an equation for the plane:

$$-4(x-1) + 2(y-0) + 0(z-2) = 0$$

or

$$\boxed{2x - y = 2}$$

(c) $\vec{r}(2) = \langle 3, 4, 4\rangle \implies (x, y, z) = (3, 4, 4)$ when $t = 2$

Distance from $(x, y, z) = (3, 4, 4)$ to $(1, 1, 1)$ is given by

$$distance = \sqrt{(3-1)^2 + (4-1)^2 + (4-1)^2} = \boxed{\sqrt{22}}$$

(ii) The missile will travel along the tangent line to the jet at the point $(3, 6, 4)$.

To find the equation of this tangent line, we need a point on the line and and a vector in the direction of the missile.

Point on the tangent line: $(3, 4, 4)$

Vector in the direction of the missile= $\mathbf{r}'(2) = \langle 1, 2, -1\rangle$

Thus, the missile will travel along the line given below for $t \geq 0$:

$$x = 3 + t$$

$$y = 4 + 2t$$

$$z = 4 - t$$

(This is one of an infinite number of possible parameterizations of this line)

This corresponds to the vector-valued function:

$$\boxed{\vec{\mathbf{r_2}}(t) = \langle 3+t, 4+2t, 4-t\rangle, \quad t \geq 0}$$

(iii) We plug in our parameterization from part (ii) to the equation of the surface:

$$z + 5 = (y-9)^2 - (x-5)^2$$

$$\implies (4-t) + 5 = (4+2t-9)^2 - (3+t-5)^2$$

$$\implies 9 - t = 4t^2 - 20t + 25 - (t^2 - 4t + 4)$$

$$\implies 3t^2 - 15t + 12 = 0$$

$$\implies 3(t-1)(t-4)=0$$

$$\implies t=1 \text{ or } t=4$$

$$\vec{\mathbf{r}}_2(1)=\langle 4,6,3\rangle \text{ and } \vec{\mathbf{r}}_2(4)=\langle 7,12,0\rangle$$

Thus the missile intersects this surface at the points

$$\boxed{(x,y,z)=(4,6,3) \textbf{ and } (7,12,0)}$$

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Question 3 (25 pts)
Consider a particle moving along a path $\vec{\mathbf{r}}(t)$ with a **constant speed** of 3 along the entire path. We also know that at the particular time $t=5$ the following is true:

$$\vec{\mathbf{r}}(5)=3\hat{\mathbf{j}}-2\hat{\mathbf{k}} \qquad \hat{\mathbf{T}}(5)=\frac{\hat{\mathbf{i}}+2\hat{\mathbf{k}}}{\sqrt{5}} \qquad \hat{\mathbf{N}}(5)=-\hat{\mathbf{j}} \qquad \kappa(5)=2$$

For each of the following quantities, determine if you have enough information to calculate the exact quantity. If so, calculate it and justify your answer. If not, explain what additional information you'd need to calculate the exact quantity.

(a) $\hat{\mathbf{B}}(5)$
(b) $\hat{\mathbf{B}}(0)$
(c) $\vec{\mathbf{v}}(5)$
(d) $\vec{\mathbf{a}}(5)$
(e) $\vec{\mathbf{v}}\cdot\vec{\mathbf{a}}$ at the time $t=10$

<u>**SOLUTION:**</u>

(a)

$$\hat{\mathbf{B}}(5)=\hat{\mathbf{T}}(5)\times\hat{\mathbf{N}}(5)=\frac{1}{\sqrt{5}}\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 1 & 0 & 2\\ 0 & -1 & 0\end{vmatrix}=\boxed{\frac{1}{\sqrt{5}}(2\hat{\mathbf{i}}-\hat{\mathbf{k}})}$$

(b) We cannot determine this vector with the information given. We would either need to be given the vector-valued function $\vec{\mathbf{r}}(t)$ or be given $\hat{\mathbf{T}}(0)$ and $\hat{\mathbf{N}}(0)$

(c)

$$\vec{\mathbf{v}}(5)=||\vec{\mathbf{r}}'(5)||\hat{\mathbf{T}}(5)$$

We're given $||\vec{\mathbf{r}}'(t)||=3$ for all t.
Thus

$$\boxed{\vec{\mathbf{v}}(5)=\frac{3}{\sqrt{5}}(\hat{\mathbf{i}}+2\hat{\mathbf{k}})}$$

(d)

$$\vec{\mathbf{a}}(5)=a_T\hat{\mathbf{T}}(5)+a_N\hat{\mathbf{N}}(5)$$

.

$a_T(5)=\frac{d}{dt}||\vec{\mathbf{r}}'(t)||=\frac{d}{dt}(3)=0$ since speed is constant.
$a_N(5)=\kappa(5)||\vec{\mathbf{r}}'(5)||^2=(2)(3)^2=18$
Thus

$$\vec{\mathbf{a}}(5)=(0)\hat{\mathbf{T}}(5)+18\hat{\mathbf{N}}(5)=\boxed{-18\hat{\mathbf{j}}}$$

.

(e) Since the speed is constant along the entire path we know

$$\vec{\mathbf{a}}(10) = (0)\hat{\mathbf{T}}(10) + a_N\hat{\mathbf{N}}(10) = a_N\hat{\mathbf{N}}(10) = 9\kappa(10)\hat{\mathbf{N}}(10)$$

$$\vec{\mathbf{v}}(10) = 3\hat{\mathbf{T}}(10)$$

.

Thus

$$\vec{\mathbf{a}}(10) \cdot \vec{\mathbf{v}}(10) = 9\kappa(10)\hat{\mathbf{N}}(10) \cdot 3\hat{\mathbf{T}}(10) = 27\kappa(10)\left(\hat{\mathbf{N}}(10) \cdot \hat{\mathbf{T}}(10)\right) = \boxed{0}$$

since $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$ are always orthogonal.

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Question 4 (23 pts)

As part of an engineering project, you are trying to weld two steel objects together. The surfaces of the two objects are given by

$$\frac{(x-1)^2}{50} + \frac{y^2}{100} = 1 \quad \textit{and} \quad z = \frac{x^2}{2} + \frac{y^2}{4}$$

where distances are measured in feet. The objects are joined where these surfaces intersect. You know that to weld the objects together, you will need 0.05 pounds of welding wire per foot of weld.

(a) Classify (i.e. give the name of) both of the surfaces.

(b) Give a parameterization that traces out the curve of intersection of the surfaces. Give a parametric interval such that this curve is traced once.

(c) If you have 3 lbs of welding wire available, will you be able to complete the weld? Show work fully justifying your answer.

SOLUTION:

(a) The first surface is an elliptic cylinder. The second surface is an elliptic paraboloid.

(b) .

From the first surface, we know that the intersection will make an ellipse if projected onto the xy-plane. Then the second surface gives the z-coordinate based on x and y. Thus, all we need to do is parameterize our ellipse and then plug in the x and y to get the z component

$$\begin{aligned}
x(t) &= \sqrt{50}\cos(t) + 1 \\
&= 5\sqrt{2}\cos(t) + 1 \\
y(t) &= \sqrt{100}\sin(t) \\
&= 10\sin(t) \\
z(t) &= \frac{(5\sqrt{2}\cos(t)+1)^2}{2} + \frac{(10\sin(t))^2}{4} \\
&= \frac{50\cos^2(t)}{2} + \frac{10\sqrt{2}\cos(t)}{2} + \frac{1}{2} + \frac{100\sin^2(t)}{4} \\
&= 5\sqrt{2}\cos(t) + \frac{51}{2} \\
\mathbf{r}(t) &= \langle 5\sqrt{2}\cos(t) + 1, 10\sin(t), 5\sqrt{2}\cos(t) + \frac{51}{2}\rangle \\
&\quad 0 \le t \le 2\pi
\end{aligned}$$

(c) .

In order to know if we have enough welding wire, we need to calculate the length of the weld, which is given by the arc length of the curve of intersection. So,

$$\begin{aligned}
\mathbf{r}'(t) &= \langle -5\sqrt{2}\sin(t), 10\cos(t), -5\sqrt{2}\sin(t)\rangle \\
|\mathbf{r}'(t)| &= \sqrt{50\sin^2(t) + 100\cos^2(t) + 50\sin^2(t)} \\
&= \sqrt{100} \\
&= 10 \\
L &= \int_0^{2\pi} 10dt \\
&= 20\pi
\end{aligned}$$

Then we take this length and multiply it by the wire requirement,

$$\text{wire needed} = 20\pi\text{ft} * 0.05\text{lbs/ft} = \pi\text{lbs}$$

Since π is greater than 3, we do not have enough wire to complete the weld.

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End Of Exam