1. [2350/072321 Exam (45 pts)] Let $\mathbf{F} = y\,\mathbf{i} + yz\,\mathbf{j} - \frac{1}{2}x^2\,\mathbf{k}$ and consider the surface, $\mathcal{S}$, given by $x^2 + y^2 - z^2 = -1, 1 \le z \le \sqrt{5}$ with upward pointing normal.

   (a) (5 pts) Name the surface.

   (b) (20 pts) Find the circulation of $\mathbf{F}$ on the boundary of $\mathcal{S}$ by direct computation. The identity $1 - 2\sin^2 t = \cos 2t$ may be helpful.

   (c) (20 pts) Find the circulation of $\mathbf{F}$ on the boundary of $\mathcal{S}$ using Stokes' Theorem.

SOLUTION:

(a) The surface is the upper branch of a hyperboloid of two sheets.

(b) The boundary of the surface is obtained by setting $z = \sqrt{5}$, that is, $x^2 + y^2 = 4$. The upward pointing normal induces a counterclockwise orientation on the boundary. Parameterizing this boundary gives

$$\mathbf{r}(t) = \langle 2\cos t, 2\sin t, \sqrt{5}\rangle \quad 0 \le t \le 2\pi$$

$$\mathbf{r}'(t) = \langle -2\sin t, 2\cos t, 0\rangle$$

$$\mathbf{F}[\mathbf{r}(t)] = 2\sin t\,\mathbf{i} + 2\sqrt{5}\sin t\,\mathbf{j} - 2\cos^2 t\,\mathbf{k}$$

$$\mathbf{F}[\mathbf{r}(t)] \cdot \mathbf{r}'(t) = \langle 2\sin t, 2\sqrt{5}\sin t, -2\cos^2 t\rangle \cdot \langle -2\sin t, 2\cos t, 0\rangle = -4\sin^2 t + 4\sqrt{5}\sin t\cos t$$

$$\text{Circulation} = \oint_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_0^{2\pi} \left(-4\sin^2 t + 4\sqrt{5}\sin t\cos t\right) \mathrm{d}t$$

$$= 2\int_0^{2\pi} \left(\cos 2t - 1 + \sqrt{5}\sin 2t\right) \mathrm{d}t = -4\pi$$

(c) Project the surface $g(x,y,z) = x^2 + y^2 - z^2$ onto the xy-plane so that $\mathbf{p} = \mathbf{k}$ and the region of integration, $\mathcal{R}$, is the disk of radius 2 centered at the origin.

$$\boldsymbol{\nabla} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & yz & -\frac{1}{2}x^2 \end{vmatrix} = \langle -y, x, -1\rangle$$

$$\nabla g = \langle 2x, 2y, -2z\rangle \quad \text{choose } -\nabla g \text{ for upward pointing normal}$$

$$|\nabla g \cdot \mathbf{p}| = 2z \quad \text{since } z > 0$$

$$\boldsymbol{\nabla} \times \mathbf{F} \cdot \frac{-\nabla g}{|\nabla g \cdot \mathbf{p}|} = \langle -y, x, -1\rangle \cdot \frac{\langle -2x, -2y, 2z\rangle}{2z} = -1$$

$$\text{Circulation} = \oint_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_{\mathcal{S}} \boldsymbol{\nabla} \times \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}S = \iint_{\mathcal{R}} -1\,\mathrm{d}A = -\text{area}(\mathcal{R}) = -4\pi$$

Remark: Since the "cap" of the hyperboloid shares the boundary with the hyperboloid, one could have used $g(x,y,z) = z = \sqrt{5}$ as the surface in place of the hyperboloid.

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2. [2350/072321 Exam (25 pts)] Let $\mathbf{F} = e^y\,\mathbf{i} + (xe^y + \sin z)\,\mathbf{j} + y\cos z\,\mathbf{k}$.

   (a) (12 pts) Show that $\mathbf{F}$ is conservative.

   (b) (13 pts) Find the work done by $\mathbf{F}$ on an object that moves from $(0,0,0)$ to $(1,-1,3)$.

SOLUTION:

(a)

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^y & xe^y + \sin z & y\cos z \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(y\cos z) - \frac{\partial}{\partial z}(xe^y + \sin z)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(e^y) - \frac{\partial}{\partial x}(y\cos z)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(xe^y + \sin z) - \frac{\partial}{\partial y}(e^y)\right]\mathbf{k}$$

$$= (\cos z - \cos z)\,\mathbf{i} + (0 - 0)\,\mathbf{j} + (e^y - e^y)\,\mathbf{k} = \mathbf{0}$$

Since the vector field is defined on all of $\mathbb{R}^3$, which is simply connected, $\mathbf{F}$ is conservative.

(b) Since $\mathbf{F}$ is conservative, a potential function, f, exists such that $\mathbf{F} = \nabla f$. Moreover, the work done by the force is independent of the path and is simply equal to the difference between the value of the potential function at the end of the path and that at the beginning. Find the potential function:

$$\frac{\partial f}{\partial x} = e^y \implies f(x,y,z) = \int e^y \, \mathrm{d}x = xe^y + g(y,z)$$

$$\frac{\partial f}{\partial y} = xe^y + g_y(y,z) = xe^y + \sin z \implies g_y(y,z) = \sin z \implies g(y,z) = \int \sin z \, \mathrm{d}y = y \sin z + h(z)$$

$$\implies f(x,y,z) = xe^y + y\sin z + h(z)$$

$$\frac{\partial f}{\partial z} = y\cos z + h'(z) = y\cos z \implies h'(z) = 0 \implies h(z) = \text{constant (which we can set to zero)}$$

giving the potential function as $f(x,y,z) = xe^y + y\sin z$. Using this we have

$$\text{Work} = \int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{(0,0,0)}^{(1,-1,3)} \nabla f \cdot \mathrm{d}\mathbf{r} = f(1,-1,3) - f(0,0,0) = e^{-1} - \sin 3$$

Remark: Since the vector field is conservative, you could compute a line integral using the line segment between the two points, but even with that simple path the line integral is ugly.

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3. [2350/072321 Exam (20 pts)] Use Green's theorem to find the outward flux of the vector field $\mathbf{F} = x^2y\,\mathbf{i} + 3xy^2\,\mathbf{j}$ through the boundary of the second quadrant portion of the circle of radius 3 centered at the origin.

SOLUTION:

The boundary, $\mathcal{C}$ consists of three piecewise smooth curves, two lines and a quarter circle surrounding the region $\mathcal{D}$. With $\mathbf{F} = \langle P, Q\rangle = \langle x^2y, 3xy^2\rangle$ we have

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 2xy + 6xy = 8xy$$

and using Green's Theorem gives

$$\text{Flux} = \int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}s = \int_{\mathcal{C}} x^2y\,\mathrm{d}y - 3xy^2\,\mathrm{d}x = \iint_{\mathcal{D}} 8xy\,\mathrm{d}A = 8\int_{-3}^{0}\int_{0}^{\sqrt{9-x^2}} xy\,\mathrm{d}A \quad \text{switch to polar coordinates}$$

$$= 8\int_{\pi/2}^{\pi}\int_0^3 r^3\cos\theta\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta = 8\int_{\pi/2}^{\pi} \left.\frac{r^4}{4}\right|_0^3 \frac{1}{2}\sin 2\theta\,\mathrm{d}\theta = \frac{81}{2}\cos 2\theta\Big|_{\pi}^{\pi/2} = -81$$

Remark: The integration in rectangular coordinates isn't too bad, making the switch to polar coordinates somewhat optional. ■

4. [2350/072321 Exam (20 pts)] Let $\mathcal{S}$ be the first octant portion of the plane with intercepts $(2,0,0)$, $(0,4,0)$ and $(0,0,1)$. Its area is $\sqrt{21}$. Using this information, find the average value of $f(x,y,z) = 1 + x$ over $\mathcal{S}$.

SOLUTION:

Since we know the intercepts of the plane, we can immediately write its equation as $\frac{x}{2} + \frac{y}{4} + z = 1$ or equivalently as $2x + y + 4z = 4$. (Note that one can also use the vectors $\langle 2, 0, -1\rangle$ and $\langle 2, -4, 0\rangle$ that lie in the plane to find its equation.) We project $\mathcal{S}$ onto the xy-plane giving $\mathbf{p} = \mathbf{k}$ and the region of integration as the triangle in the first octant with vertices $(0,0)$, $(2,0)$ and $(0,4)$.

$$g(x,y,z) = 2x + y + 4z$$

$$\nabla g = \langle 2, 1, 4\rangle$$

$$\|\nabla g\| = \sqrt{21}$$

$$|\nabla g \cdot \mathbf{p}| = 4$$

$$\iint_{\mathcal{S}} (1+x)\,\mathrm{d}S = \int_0^2\int_0^{4-2x} (1+x)\frac{\sqrt{21}}{4}\,\mathrm{d}y\,\mathrm{d}x = \frac{\sqrt{21}}{4}\int_0^2 (1+x)(4-2x)\,\mathrm{d}x = \frac{5\sqrt{21}}{3}$$

The average value we seek is thus $\frac{5\sqrt{21}}{3}/\sqrt{21} = \frac{5}{3}$. ■

5. [2350/072321 Exam (20 pts)] A piece of wire is in the shape of (e^t, t^2) with its left end at the point $(1, 0)$. The charge density on the wire is $q(x,y) = \dfrac{3y}{\sqrt{x^2+4y}}$ Coulombs per meter. If the total charge on the wire is 8 Coulombs, find the coordinates of the right end of the wire.

SOLUTION:

We need to compute the scalar line integral

$$\text{Total Charge} = \int_{\mathcal{C}} q(x,y)\,\mathrm{d}s$$

The left point of the wire occurs when $t = 0$ and we need to find the upper bound, say b, for t which we then can use to find the coordinates of the other end of the wire. The curve (wire) is

$$\begin{aligned} \mathbf{r}(t) &= \left\langle e^t, t^2 \right\rangle, \quad 0 \le t \le b \\ \mathbf{r}'(t) &= \left\langle e^t, 2t \right\rangle \\ \|\mathbf{r}'(t)\| &= \sqrt{e^{2t} + 4t^2} \\ q[\mathbf{r}(t)] &= \frac{3t^2}{\sqrt{e^{2t}+4t^2}} \end{aligned}$$

We then have

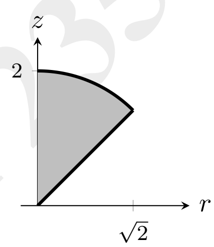
$$8 = \int_{\mathcal{C}} \frac{3y}{\sqrt{x^2+4y}}\,\mathrm{d}s = \int_0^b q[\mathbf{r}(t)]\,\|\mathbf{r}'(t)\|\,\mathrm{d}t = \int_0^b 3t^2\,\mathrm{d}t = b^3 \implies b = 2$$

giving the coordinates of the right end of the wire as $\mathbf{r}(2) = (e^2, 4)$. ■

6. [2350/072321 Exam (20 pts)] Use Gauss' Theorem to find the outward flux of the vector field $\mathbf{F} = x^3\,\mathbf{i} + y^3\,\mathbf{j} + z^3\,\mathbf{k}$ through the boundary of the solid region above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = 4$.

SOLUTION:

Let $\mathcal{E}$ be the region over whose boundary $(\mathcal{S})$ we wish to compute the flux of $\mathbf{F}$. A sketch of the region in the rz-plane (constant θ):



$$\nabla \cdot \mathbf{F} = 3\left(x^2+y^2+z^2\right)$$

Using Gauss' Theorem we have

$$\begin{aligned} \iint_{\mathcal{S}} \mathbf{F}\cdot\mathrm{d}\mathbf{S} &= \iiint_{\mathcal{E}} \nabla\cdot\mathbf{F}\,\mathrm{d}V \\ &= \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} 3\left(x^2+y^2+z^2\right)\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}z \quad \text{rectangular coordinates} \\ &= \int_0^{2\pi} \int_0^{\sqrt{2}} \int_r^{\sqrt{4-r^2}} 3\left(r^2+z^2\right) r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta \quad \text{cylindrical coordinates} \\ &= \int_0^{2\pi} \int_0^{\pi/4} \int_0^2 3\left(\rho^2\right)\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta \quad \text{spherical coordinates} \\ &= \frac{96\pi}{5}\left(2-\sqrt{2}\right) \end{aligned}$$

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