1. [2350/070921 Exam (15 pts)] Let $\mathcal{R}$ be the parallelogram in the $xy$-plane enclosed by the lines $x + 4y = 4, x + 4y = 9, x - y = 1$ and $x - y = 4$. Use a change of variables to find the volume of the solid above the region $\mathcal{R}$ and below the surface $z = 5\sqrt{(x-y)(x+4y)}$.

**SOLUTION:**

To find the requested volume, we need to compute $\iint_{\mathcal{R}} 5\sqrt{(x-y)(x+4y)}\,\mathrm{d}A$. Both the integrand and the region suggest the change of variables $u = x + 4y$ and $v = x - y$. The region $\mathcal{R}$ is given by $4 \le x + 4y \le 9$ and $1 \le x - y \le 4$, which gives the new region of integration as $4 \le u \le 9$ and $1 \le v \le 4$.

Now $u - v = 5y \Rightarrow y = \frac{1}{5}(u - v)$ so that $x = v + y = \frac{1}{5}(u + 4v)$ and

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{5} & \frac{4}{5} \\ \frac{1}{5} & -\frac{1}{5} \end{vmatrix} = -\frac{1}{5} \quad \text{and} \quad f(u,v) = 5\sqrt{uv} = 5u^{1/2}v^{1/2}$$

$$\text{Volume} = \iint_{\mathcal{R}} 5\sqrt{(x-y)(x+4y)}\,\mathrm{d}A = \int_1^4 \int_4^9 5u^{1/2}v^{1/2}\left|-\frac{1}{5}\right| \mathrm{d}u\,\mathrm{d}v = \left(\frac{2}{3}u^{3/2}\Big|_4^9\right)\left(\frac{2}{3}v^{3/2}\Big|_1^4\right) = \frac{532}{9}$$

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2. [2350/070921 Exam (15 pts)] A metal pipe with inner diameter $d_i$, outer diameter $d_o$ and length $l$ has a mass density that varies inversely with the cube of the distance from the axis of the pipe, that is, mass density, $\delta = k/\text{distance}^3$. Find the constant of proportionality, $k$, in terms of the other variables, if the total mass of the pipe is $M$. (Recall that the total mass of a three-dimensional solid is given by the triple integral of the mass density over the region occupied by the solid.) Hint: Place the axis of the pipe along the $z$-axis and pick a coordinate system that simplifies the problem.

**SOLUTION:**

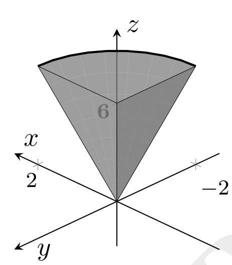
Place the $z$-axis along the axis of the pipe and use cylindrical coordinates. Then the density can be written as $\delta(r,\theta,z) = \frac{k}{r^3}$.

$$\text{Total mass} = M = \iiint\limits_{\text{pipe}} \delta\,\mathrm{d}V = \int_0^l \int_0^{2\pi} \int_{d_i/2}^{d_o/2} \frac{k}{r^3}\, r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z = \left(\int_0^l \mathrm{d}z\right)\left(\int_0^{2\pi} \mathrm{d}\theta\right)\left(k\int_{d_i/2}^{d_o/2} r^{-2}\,\mathrm{d}r\right)$$

$$= 2\pi kl \left(\frac{1}{r}\Big|_{d_o/2}^{d_i/2}\right) = 4\pi kl\left(\frac{1}{d_i} - \frac{1}{d_o}\right) \implies k = \frac{M d_i d_o}{4\pi l(d_o - d_i)}$$

■

3. [2350/070921 Exam (18 pts)] Consider the solid shown below, which is a portion of the cone $x^2 + y^2 - \left(\frac{z}{3}\right)^2 = 0$.




Each of the following triple integrals can be used to compute the volume of $\mathcal{W}$. Copy each them onto your paper and provide the six (6) appropriate limits for each one, using the given order of integration. **Do not evaluate** any of the integrals. To receive full credit, you must use the correct bounds for the figure as shown (study it carefully), not bounds for an equivalent solid in a different octant.

(a) Volume ($\mathcal{W}$) = $\int\int\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$

(b) Volume ($\mathcal{W}$) = $\int\int\int r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$

(c) Volume ($\mathcal{W}$) = $\int\int\int \rho^2 \sin\phi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi$

**SOLUTION:**

$\mathcal{W}$ is the "inside" of the portion of the cone of radius 2 and height 6 with vertex at the origin that resides above Quadrant IV.

(a) Arbitrary $y$, $z$: $x$ enters region at 0 and exits at $\sqrt{z^2/9 - y^2}$. Project onto $yz$-plane.
Arbitrary $z$: $y$ enters projected region at $-z/3$ and exits at 0. Then sum up $z$ from 0 to 6.

$$\text{Volume}(\mathcal{W}) = \int_0^6 \int_{-z/3}^0 \int_0^{\sqrt{z^2/9-y^2}} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

(b) Arbitrary $r$, $\theta$: $z$ enters region at $3r$ and exits at 6. Project onto the $xy$-plane.

Arbitrary $\theta$: $r$ enters projected region at 0 and exits at 2. Then sum up $\theta$ from $3\pi/2$ to $2\pi$.

$$\text{Volume}(\mathcal{W}) = \int_{3\pi/2}^{2\pi} \int_0^2 \int_{3r}^6 r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$$

(c) Arbitrary $\theta$, $\phi$: $\rho$ enters the region at 0 and exits at $6\sec\phi$. These rays are then summed from $\theta = 3\pi/2$ to $\theta = 2\pi$ and swept from $\phi = 0$ to $\phi = \tan^{-1}\frac{1}{3}$.

$$\int_0^{\tan^{-1}\left(\frac{1}{3}\right)} \int_{3\pi/2}^{2\pi} \int_0^{6\sec\phi} \rho^2 \sin\phi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi$$

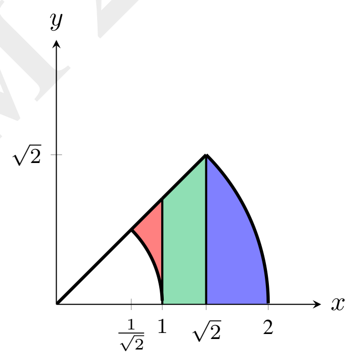
■

4. [2350/070921 Exam (22 pts)] Use polar coordinates to combine the following into a single double integral and then evaluate the resulting polar coordinate double integral. Making a sketch should prove beneficial.

$$\int_{1/\sqrt{2}}^1 \int_{\sqrt{1-x^2}}^x xy\,\mathrm{d}y\,\mathrm{d}x + \int_1^{\sqrt{2}} \int_0^x xy\,\mathrm{d}y\,\mathrm{d}x + \int_{\sqrt{2}}^2 \int_0^{\sqrt{4-x^2}} xy\,\mathrm{d}y\,\mathrm{d}x$$

**SOLUTION:**

Here is a sketch of the region of integration, based on the three given integrals.




Converting to polar coordinates yields:

$$\int_{1/\sqrt{2}}^1 \int_{\sqrt{1-x^2}}^x xy\,\mathrm{d}y\,\mathrm{d}x + \int_1^{\sqrt{2}} \int_0^x xy\,\mathrm{d}y\,\mathrm{d}x + \int_{\sqrt{2}}^2 \int_0^{\sqrt{4-x^2}} xy\,\mathrm{d}y\,\mathrm{d}x$$

$$= \int_0^{\pi/4} \int_1^2 (r\cos\theta)(r\sin\theta)\,r\,\mathrm{d}r\,\mathrm{d}\theta = \left(\int_0^{\pi/4} \frac{1}{2}\sin 2\theta\,\mathrm{d}\theta\right)\left(\int_1^2 r^3\,\mathrm{d}r\right) = \frac{15}{16}$$

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5. [2350/070921 Exam (15 pts)] Evaluate $\displaystyle\int_0^4 \int_0^1 \int_{2y}^2 \frac{2\cos(x^2)}{\sqrt{z}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$. Hint: The antiderivative of $\cos(x^2)$ is not $-\sin\left(x^2\right)$.

**SOLUTION:**

Integrate with respect to $y$ first since $\cos(x^2)$ does not possess an antiderivative that is an elementary function.

$$\begin{aligned}\int_0^4\int_0^1\int_{2y}^2 \frac{2\cos(x^2)}{\sqrt{z}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \left(2\int_0^4 z^{-1/2}\,\mathrm{d}z\right)\int_0^2\int_0^{x/2}\cos(x^2)\,\mathrm{d}y\,\mathrm{d}x\\ &= 4\sqrt{z}\,\Big|_0^4 \int_0^2 \cos(x^2)\;y\,\Big|_0^{x/2}\,\mathrm{d}x\\ &= 4\int_0^2 x\cos(x^2)\,\mathrm{d}x\\ &\overset{u=x^2}{=} 2\int_0^4 \cos u\,\mathrm{d}u\\ &= 2\sin 4\end{aligned}$$

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6. [2350/070921 Exam (15 pts)] Using spherical coordinates with $\mathrm{d}V = \rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$, set up, but **do not evaluate**,

$$\iiint_Q \frac{z}{\sqrt{1+x^2+y^2}}\,\mathrm{d}V,$$

where $Q$ is the region of the sphere $x^2+y^2+z^2=4$ below the plane $z=-\sqrt{3}$ and under the first octant.

**SOLUTION:**

The integrand becomes $\dfrac{z}{\sqrt{1+x^2+y^2}} = \dfrac{\rho\cos\phi}{\sqrt{1+\rho^2\sin^2\phi}}$. The region of integration is a spherical cap described by the inequalities $-\sqrt{3}\sec\phi \le \rho \le 2$, $5\pi/6 \le \phi \le \pi$, and $0 \le \theta \le \pi/2$. Thus

$$\iiint_Q \frac{z}{\sqrt{1+x^2+y^2}}\,\mathrm{d}V = \int_0^{\pi/2}\int_{5\pi/6}^{\pi}\int_{-\sqrt{3}\sec\phi}^{2} \frac{\rho^3\sin\phi\cos\phi}{\sqrt{1+\rho^2\sin^2\phi}}\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$$

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