1. [2350/062521 Exam (15 pts)] Consider the function $z = f(x, y) = 25x^2 + 4y^2 + 4$. On your paper write the letters (a)-(e) and next to each one write the word TRUE or FALSE as appropriate. No work is required and no partial credit will be given.

   (a) The level curves of $f(x, y)$ are circles.

   (b) The tangent plane to the function at $(x, y) = (0, 0)$ is horizontal.

   (c) The vertical trace of the function in the plane $y = 2$ is an ellipse.

   (d) If you were to walk along the curve $(2\cos t, 5\sin t)$ in the xy-plane, the height of the surface above you would be constant.

   (e) The domain and range of the function are, respectively, $\mathbb{R}^2$ and $[0, \infty)$.

SOLUTION:

(a) **FALSE**; they are ellipses

(b) **TRUE**; the tangent plane at the point $(0, 0)$ is $z = 4$

(c) **FALSE**; it is a parabola

(d) **TRUE**; the curve is the level curve corresponding to $z = 100$

(e) **FALSE**; the range is $[4, \infty)$

■

2. [2350/062521 Exam (30 pts)] Parts (a) and (b) are not related.

   (a) (18 pts) A certain portion of a forest consists of its boundary, given by the lines $|x| = 5, |y| = 5$, and the region inside the boundary. The elevation in the forest is $h(x, y) = 3xy - x^3 - y^3 + 2$.

      i. (4 pts) A friend of yours (as well as your grader) does not want numbers or lengthy calculations, just a simple yes or no answer, with a verbal mathematical justification, to the question "Is there a highest and lowest point in the forest?"

      ii. (8 pts) Are there any saddles (passes) or local hills or valleys inside the boundaries of the forest? If so, find their locations and their elevations. If there are none, explain why not.

      iii. (6 pts) You and your friend are hiking along a trail whose projection onto the xy-plane is given by $(x(t), y(t)) = \left(t, \frac{1}{2}t^2\right)$.

         A. (2 pts) What is your elevation when $t = 2$?

         B. (4 pts) Use the chain rule to determine whether your elevation is increasing or decreasing when $t = 2$.

   (b) (12 pts) You are making a rectangular chicken run using 12 feet of fencing for a boundary. Use Lagrange Multipliers to find the dimensions of the run that will give the chickens the most area in which to frolic.

SOLUTION:

(a) i. Yes. The elevation function is continuous and the region of concern (the forest) is closed and bounded. The Extreme Value Theorem guarantees that $h(x, y)$ will attain an absolute maximum and minimum value in the forest so there will be a highest and lowest point in the forest.

ii.

$$h_x = 3y - 3x^2 = 0 \implies y = x^2$$
$$h_y = 3x - 3y^2 = 0 \implies x = y^2$$

Substituting the second equation on the right into the first yields $y = y^4 \implies y(1 - y^3) = 0 \implies y = 0, 1$, leading to the critical points $(0, 0)$ and $(1, 1)$. Now apply the second derivatives test.

$$h_{xx} = -6x, h_{yy} = -6y, h_{xy} = 3 \implies D(x, y) = (-6x)(-6y) - 3^2 = 36xy - 9$$

$$D(0, 0) = -9 \implies (0, 0) \text{ is a saddle with elevation } h = 2$$

$$D(1, 1) = 27, h_{xx}(1, 1) = -6 \implies (1, 1) \text{ is a local maximum (hill) with elevation } h = 3$$

iii. A. When $t = 2$ we have $(x(2), y(2)) = (2, 2)$ and $h(2, 2) = -2$

B.

$$\frac{dh}{dt} = \frac{\partial h}{\partial x}\frac{dx}{dt} + \frac{\partial h}{\partial y}\frac{dy}{dt} = \left(3y - 3x^2\right)(1) + \left(3x - 3y^2\right)(t)$$

$$\left.\frac{dh}{dt}\right|_{t=2} = \left[3(2) - 3\left(2^2\right)\right] + \left[3(2) - 3\left(2^2\right)\right](2) = -18 \quad \text{elevation is decreasing}$$

(b) Let the sides of the run be x and y. Then we need to maximize the area function $f(x,y) = xy$ subject to the constraint $g(x,y) = 2x + 2y = 12$.

$$\begin{aligned} f_x = y \quad g_x = 2 &\implies y = 2\lambda \\ f_y = x \quad g_y = 2 &\implies x = 2\lambda \end{aligned}$$

Together, these give $y = x$, which when used in the constraint yields $2x + 2x = 12 \implies x = 3$. The dimensions of the chicken run should be 3 feet by 3 feet.

■

3. [2350/062521 Exam (15 pts)] Parts (a) and (b) are not related.

(a) (6 pts) Find the following limits or show that they do not exist.

i. $\displaystyle\lim_{(x,y,z)\to(\frac{\pi}{3},\frac{\pi}{6},\frac{\pi}{4})} \frac{\cos x + \sin y + \tan z}{\sqrt{3x + 6y + 4z}}$ ii. $\displaystyle\lim_{(x,y)\to(1,0)} \frac{1-x}{x+y-1}$

(b) (9 pts) The mass of a certain object is given by the function $m = \frac{2}{3}\pi l^3 w^{3/2}$ where l is the length of the object and w is its width. Consider two such objects, one with a length of 1 unit and a width of 4 units and another with a length of 4 units and a width of 1 unit. Which object's mass is more sensitive to a small change in width? Justify your answer using mathematical techniques learned in this course.

SOLUTION:

(a) i. Direct substitution yields

$$\lim_{(x,y,z)\to(\frac{\pi}{3},\frac{\pi}{6},\frac{\pi}{4})} \frac{\cos x + \sin y + \tan z}{\sqrt{3x + 6y + 4z}} = \frac{\cos(\pi/3) + \sin(\pi/6) + \tan(\pi/4)}{\sqrt{3(\pi/3) + 6(\pi/6) + 4(\pi/4)}} = \frac{1/2 + 1/2 + 1}{\sqrt{3\pi}} = \frac{2}{\sqrt{3\pi}} = \frac{2\sqrt{3\pi}}{3\pi}$$

ii. Direct substitution yields the indeterminate form $0/0$ requiring further analysis. Consider approaching the point $(1,0)$ along the line $x = 1$. In this case we have

$$\lim_{(1,y)\to(1,0)} \frac{1-1}{1+y-1} = \lim_{(1,y)\to(1,0)} 0 = 0$$

On the other hand, if we approach $(1,0)$ along the line $y = 0$ we have

$$\lim_{(x,0)\to(1,0)} \frac{x-1}{x+0-1} = \lim_{(x,0)\to(1,0)} 1 = 1$$

Since two different paths to $(1,0)$ result in two different limits, the limit does not exist.

(b) The total differential for the mass is

$$dm = \frac{\partial m}{\partial l}\,dl + \frac{\partial m}{\partial w}\,dw = 2\pi l^2 w^{3/2}\,dl + \pi l^3 w^{1/2}\,dw$$

Evaluating this at the various points gives

$$\begin{aligned} l = 1, w = 4 &\implies dm = 16\pi\,dl + 2\pi\,dw \\ l = 4, w = 1 &\implies dm = 32\pi\,dl + 64\pi\,dw \to \text{ more sensitive to small changes in width } w \end{aligned}$$

■

4. [2350/062521 Exam (14 pts)] Jack and Jill went up a hill to the point (x_0, y_0, z_0) and got caught there in a lightning storm. In a fit of panic Jack darted off in the direction of $\mathbf{A} = 2\mathbf{i} - \mathbf{j}$ and noted at that instant that his altitude was changing at an instantaneous rate of $-\sqrt{5}/5$ ft/ft. Panic stricken as well, Jill ran in the direction of $\mathbf{B} = -3\mathbf{i} + \mathbf{j}$, noting an instantaneous rate of change of altitude of $-\sqrt{10}/5$ ft/ft. In what direction should they have run in order to start descending the hill at the fastest instantaneous rate? What would that rate have been?

SOLUTION:

We are given the value of the directional derivative for both Jack and Jill as well as their directions of movement. We need to find the gradient vector at the point (x_0, y_0).

Jack:

$$\begin{aligned} D_{\mathbf{u}} f(x_0,y_0) = \nabla f(x_0,y_0)\cdot\mathbf{u} &= \langle f_x(x_0,y_0), f_y(x_0,y_0)\rangle \cdot \frac{\mathbf{A}}{\|\mathbf{A}\|} \\ &= \langle f_x(x_0,y_0), f_y(x_0,y_0)\rangle \cdot \frac{1}{\sqrt{5}}\langle 2,-1\rangle \\ &= \frac{2}{\sqrt{5}} f_x(x_0,y_0) - \frac{1}{\sqrt{5}} f_y(x_0,y_0) = -\frac{\sqrt{5}}{5} \\ &\implies 2f_x(x_0,y_0) - f_y(x_0,y_0) = -1 \end{aligned}$$

Jill:

$$\begin{aligned} D_{\mathbf{u}}f(x_0,y_0) &= \nabla f(x_0,y_0)\cdot\mathbf{u} = \langle f_x(x_0,y_0), f_y(x_0,y_0)\rangle \cdot \frac{\mathbf{B}}{\|\mathbf{B}\|} \\ &= \langle f_x(x_0,y_0), f_y(x_0,y_0)\rangle \cdot \frac{1}{\sqrt{10}}\langle -3, 1\rangle \\ &= -\frac{3}{\sqrt{10}}f_x(x_0,y_0) + \frac{1}{\sqrt{10}}f_y(x_0,y_0) = -\frac{\sqrt{10}}{5} \\ &\implies -3f_x(x_0,y_0) + f_y(x_0,y_0) = -2 \end{aligned}$$

Solve the linear system

$$\begin{aligned} 2f_x(x_0,y_0) - f_y(x_0,y_0) &= -1 \\ -3f_x(x_0,y_0) + f_y(x_0,y_0) &= -2 \end{aligned}$$

to find $\nabla f(x_0,y_0) = 3\mathbf{i} + 7\mathbf{j}$. So to instantaneously descend the quickest Jack and Jill should run in the direction of

$$-\nabla f(x_0,y_0) = -3\mathbf{i} - 7\mathbf{j}$$

Their instantaneous rate of descent in this case is $-\|-\nabla f(x_0,y_0)\| = -\sqrt{58}$ ft/ft (or $\sqrt{58}$ ft/ft down). ■

5. [2350/062521 Exam (26 pts)] Parts (a) and (b) are not related.

   (a) (10 pts) Consider the function $g(x,y,z,t)$ where

$$x = u^2 + v, \quad y = u + v^2, \quad z = \ln(v/u), \quad t = e^{uv}$$

   Suppose when $z = -\ln 4$ and $v = 1$ that $g_x = 2$, $g_y = -3$, $g_z = 6$ and $g_t = -2$. Calculate the instantaneous rate of change of $g(x,y,z,t)$ with respect to u at this point.

   (b) (16 pts) Consider the function $f(x,y) = \ln(xy)$.

      i. (8 pts) Calculate the first order Taylor (tangent plane) approximation (linearization) of f about the point $(1,1)$.

      ii. (2 pts) Approximate the value of $f(1.1, 1.2)$ using your first order Taylor polynomial.

      iii. (6 pts) Find an upper bound on the error in your first order Taylor approximation over the region where $|x-1| \le 0.1$ and $|y-1| \le 0.2$.

SOLUTION:

(a) Note that with $v = 1$, $z = -\ln 4 = \ln\frac{1}{4} = \ln\frac{1}{u} \implies u = 4$

$$\begin{aligned} \frac{\partial g}{\partial u} &= \frac{\partial g}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial g}{\partial z}\frac{\partial z}{\partial u} + \frac{\partial g}{\partial t}\frac{\partial t}{\partial u} \\ &= \frac{\partial g}{\partial x}(2u) + \frac{\partial g}{\partial y}(1) + \frac{\partial g}{\partial z}\left(\frac{u}{v}\right)\left(-\frac{v}{u^2}\right) + \frac{\partial g}{\partial t}ve^{uv} \\ &= \frac{\partial g}{\partial x}(2u) + \frac{\partial g}{\partial y} - \frac{\partial g}{\partial z}\left(\frac{1}{u}\right) + \frac{\partial g}{\partial t}ve^{uv} \\ \left.\frac{\partial g}{\partial u}\right|_{(u,v)=(4,1)} &= 2(8) - 3 - 6\left(\frac{1}{4}\right) - 2e^4 \\ &= \frac{23}{2} - 2e^4 \end{aligned}$$

(b) i. The first order Taylor approximation to f about $(1,1)$ is

$$L(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1).$$

We need to compute the first order partial derivatives and evaluate f, f_x, and f_y at $(1,1)$.

$$\begin{aligned} f(x,y) &= \ln(x) + \ln(y) && \implies f(1,1) = 0 \\ f_x(x,y) &= \frac{1}{x} && \implies f_x(1,1) = 1 \\ f_y(x,y) &= \frac{1}{y} && \implies f_y(1,1) = 1 \end{aligned}$$

which gives

$$L(x,y) = 0 + 1(x-1) + 1(y-1) = x + y - 2.$$

ii. We have $f(1.1, 1.2) \approx L(1.1, 1.2)$ or

$$f(1.1, 1.2) \approx 1.1 + 1.2 - 2 = 0.3.$$

iii. The error in the first order Taylor approximation is given by

$$|E(x, y)| \le \frac{M}{2}(|x-1| + |y-1|)^2$$

where M is an upper bound on the absolute values of the second partial derivatives in the region of interest.

$$\begin{aligned} f_{xx}(x,y) &= -\frac{1}{x^2} && \implies |f_{xx}| = \left|-\frac{1}{x^2}\right| \le \frac{1}{0.9^2} = \frac{100}{81} \\ f_{yy}(x,y) &= -\frac{1}{y^2} && \implies |f_{yy}| = \left|-\frac{1}{y^2}\right| \le \frac{1}{0.8^2} = \frac{100}{64} = \frac{25}{16} \\ f_{xy}(x,y) &= 0 && \implies |f_{xy}| = 0 \end{aligned}$$

We can take $M = \frac{25}{16}$. Then, in the region where $|x-1| \le 0.1$ and $|y-1| \le 0.2$,

$$|E(x,y)| \le \frac{25}{32}(0.1+0.2)^2 = \frac{25}{32}\left(\frac{3}{10}\right)^2 = \frac{25}{32}\left(\frac{9}{100}\right) = \frac{9}{128}.$$

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