1. [APPM 2350 Exam (12 pts)] On your paper write the letters (a)-(f) and next to each one write the word TRUE or FALSE as appropriate. No work is required and no partial credit will be given.

   (a) $(\mathbf{u}\cdot\mathbf{v})\times\mathbf{w}=\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$

   (b) The cross product of two nonzero vectors that are scalar multiples of each other has magnitude 0.

   (c) $(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})=\|\mathbf{u}\|^2+\|\mathbf{v}\|^2$

   (d) The intersection of the plane $z=2$ and the surface $4x^2+y^2+4z^2-4y-24z+36=0$ is an ellipse.

   (e) The normal component of the acceleration of a particle moving along a straight line is always zero.

   (f) The unit binormal vector $\mathbf{B}$ for a curve lying in the plane $z=3$ is $\pm\mathbf{k}$.

   SOLUTION:

   (a) **FALSE**; cross product of a scalar and a vector is not defined

   (b) **TRUE**; vectors that are scalar multiples of one another are parallel so the angle θ between them is zero

   $$\|\mathbf{a}\times\mathbf{b}\|=\|\mathbf{a}\|\|\mathbf{b}\|\sin\theta=0$$

   (c) **FALSE**;

   $$(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})=\mathbf{u}\cdot\mathbf{u}-\mathbf{u}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{u}-\mathbf{v}\cdot\mathbf{v}=\|\mathbf{u}\|^2-\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{v}-\|\mathbf{v}\|^2=\|\mathbf{u}\|^2-\|\mathbf{v}\|^2$$

   (d) **FALSE**; Completing the square in the original equation of the surface yields

   $$x^2+\left[\frac{(y-2)}{2}\right]^2+(z-3)^2=1$$

   showing that the surface is an ellipsoid centered at $(0,2,3)$. If $z=2$, $x^2+\left[\frac{(y-2)}{2}\right]^2=0$, the only solution of which is $x=0$, $y=2$. Thus the trace is the single point $(0,2,2)$.

   (e) **TRUE**; The curvature of a straight line is 0 and since the normal component of the acceleration is proportional to the curvature it, too, is 0.

   (f) **TRUE**; $\mathbf{T}$ and $\mathbf{N}$ lie in the plane $z=3$ and thus have only $\mathbf{i}$ and $\mathbf{j}$ components. Their cross product will then only have a $\mathbf{k}$ component, which is a unit vector.

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2. [APPM 2350 Exam (16 pts)] A particle travels along the helix given by $\mathbf{r}(t)=\cos t\,\mathbf{i}+\sin t\,\mathbf{j}+t\,\mathbf{k}$. At time $t=\pi$ the particle leaves the path and flies off on a tangent. Find the location of the particle at $t=2\pi$ assuming no forces act on it after it leaves the helix.

   SOLUTION:

   The tangent vector to the helix is $\mathbf{r}'(t)=-\sin t\,\mathbf{i}+\cos t\,\mathbf{j}+\mathbf{k}$. The particle leaves the helix at $\mathbf{r}(\pi)=-\mathbf{i}+\pi\,\mathbf{k}$ and moves in the direction of the tangent vector at this point, namely $\mathbf{r}'(\pi)=-\mathbf{j}+\mathbf{k}$. Since no other forces are acting on the particle, it moves in this direction for $t\geq\pi$. This trajectory is a straight line through the point $(-1,0,\pi)$ in the direction of $-\mathbf{j}+\mathbf{k}$. The equation of this line is $\mathbf{L}(t)=\langle-1,0,\pi\rangle+(t-\pi)\langle0,-1,1\rangle$ for $t\geq\pi$. When $t=2\pi$, the particle is at $\mathbf{L}(2\pi)=\langle-1,0,\pi\rangle+(2\pi-\pi)\langle0,-1,1\rangle=\langle-1,-\pi,2\pi\rangle$. ■

3. [APPM 2350 Exam (10 pts)] Find the equation of, and identify, the quadric surface whose points are equidistant from the point $P_0(2,0,0)$ and the plane containing the point $(-2,0,0)$ whose normal vector is $\mathbf{i}$.

   SOLUTION:

   The equation of the plane is $x=-2$. Let $P(x,y,z)$ be a point on the surface. The distant from P to the plane is $\sqrt{(x+2)^2}$ and the distance from P to P_0 is $\sqrt{(x-2)^2+y^2+z^2}$. Equating these two distances gives

   $$\sqrt{(x+2)^2}=\sqrt{(x-2)^2+y^2+z^2}$$
   $$x^2+4x+4=x^2-4x+4+y^2+z^2$$
   $$x=\frac{1}{8}\left(y^2+z^2\right)$$

   The surface is a (circular) paraboloid. ■

4. [APPM 2350 Exam (28 pts)] The following problems are not related.

(a) (8 pts) Consider the vector function $\mathbf{r}(t) = \langle t^2, \sin t - t\cos t, \cos t + t \sin t\rangle$ for $0 \le t \le c$. Find the value of c for which the arc length is $8\sqrt{5}$.

(b) (20 pts) Consider the vector function $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$ with $-\infty < t < \infty$.

i. (10 pts) Compute the *torsion*, τ (the measure of the degree of twisting of a curve), given by $\tau = \dfrac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{\|\mathbf{r}' \times \mathbf{r}''\|^2}$ at the point $(2, 4, 8)$.

ii. (10 pts) Are there any points on the curve where the velocity and acceleration vectors are orthogonal? If so, find them. If not, explain why not.

SOLUTION:

(a) Need to find c such that $s(c) = \displaystyle\int_0^c \|\mathbf{r}'(u)\|\,\mathrm{d}u$.

$$\|\mathbf{r}'(t)\| = \|\langle 2t, t\sin t, t\cos t\rangle\| = \sqrt{4t^2 + t^2\sin^2 t + t^2\cos^2 t} = \sqrt{5t^2} = \sqrt{5}|t| = \sqrt{5}t \text{ since } t \ge 0$$

Therefore

$$\int_0^c \sqrt{5}t\,\mathrm{d}t = 8\sqrt{5} \implies \left.\frac{\sqrt{5}t^2}{2}\right|_0^c = \frac{\sqrt{5}c^2}{2} = 8\sqrt{5} \implies c = 4 \quad (\text{need } c > 0)$$

(b) i. $\mathbf{r}'(t) = \langle 1, 2t, 3t^2\rangle$ and $\mathbf{r}''(t) = \langle 0, 2, 6t\rangle$ and $\mathbf{r}'''(t) = \langle 0, 0, 6\rangle$. The curve passes through the point $(2, 4, 8)$ when $t = 2$ so that

$$\mathbf{r}'(2) \times \mathbf{r}''(2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 12 \\ 0 & 2 & 12 \end{vmatrix} = \langle 24, -12, 2\rangle \implies \|\mathbf{r}'(2) \times \mathbf{r}''(2)\| = \sqrt{24^2 + (-12)^2 + 2^2} = \sqrt{724}$$

$$(\mathbf{r}'(2) \times \mathbf{r}''(2)) \cdot \mathbf{r}'''(2) = \langle 24, -12, 2\rangle \cdot \langle 0, 0, 6\rangle = 12 \implies \tau = \frac{12}{724} = \frac{3}{181}$$

ii. We need to determine if there are any values of t where $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = 0$.

$$\mathbf{r}'(t) \cdot \mathbf{r}''(t) = \langle 1, 2t, 3t^2\rangle \cdot \langle 0, 2, 6t\rangle = 4t + 18t^3 = 2t(2 + 9t^2) = 0 \implies t = 0$$

The velocity and acceleration vectors are perpendicular when $t = 0$ which is the point $(0, 0, 0)$ on the curve.

5. [APPM 2350 Exam (34 pts)] Parts (a) and (b) are not related.

(a) (10 pts) Find the position vector $\mathbf{r}(t)$ of an object subject to the following conditions: it undergoes an acceleration of $e^t\,\mathbf{i} + 2t\,\mathbf{j} + (t+1)\,\mathbf{k}$ for $t \ge 0$ and it begins its motion at $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ with a velocity of $\mathbf{i} + \mathbf{k}$.

(b) (24 pts) Consider the intersecting lines $L_1(t) = \langle 7 - 2t, t, -4 - t\rangle$ and $L_2(s) = \langle 3 + 2s, -3 + 4s, -8 + 3s\rangle$.

i. (6 pts) Find the coordinates of the point where the lines intersect.

ii. (6 pts) Find the equation of the plane containing the lines. Write your final answer in the form $az + by + cz = d$.

iii. (6 pts) Find the symmetric equations of the line normal to the plane you found in part (ii) and passing through the point you found in part (i).

iv. (6 pts) Find the coordinates of the point where the line from part (iii) intersects the plane $x + y + z = 2$.

SOLUTION:

(a) Integrate the acceleration to find that

$$\mathbf{v}(t) = \int \mathbf{a}(t)\,\mathrm{d}t = \int \mathbf{r}''(t)\,\mathrm{d}t = \mathbf{r}'(t) = \int \langle e^t, 2t, t+1\rangle\,\mathrm{d}t = \left\langle e^t, t^2, \frac{1}{2}t^2 + t\right\rangle + \mathbf{C}$$

Then $\mathbf{r}'(0) = \langle 1, 0, 1\rangle = \langle 1, 0, 0\rangle + \mathbf{C} \implies \mathbf{C} = \langle 0, 0, 1\rangle \implies \mathbf{r}'(t) = \left\langle e^t, t^2, \frac{1}{2}t^2 + t + 1\right\rangle$. Integrate the velocity to find that

$$\mathbf{r}(t) = \int \mathbf{v}(t)\,\mathrm{d}t = \int \mathbf{r}'(t)\,\mathrm{d}t = \int \left\langle e^t, t^2, \frac{1}{2}t^2 + t + 1\right\rangle\,\mathrm{d}t = \left\langle e^t, \frac{1}{3}t^3, \frac{1}{6}t^3 + \frac{1}{2}t^2 + t\right\rangle + \mathbf{C}$$

Then $\mathbf{r}(0) = \langle 1, 2, 2\rangle = \langle 1, 0, 0\rangle + \mathbf{C} \implies \mathbf{C} = \langle 0, 2, 2\rangle \implies \mathbf{r}(t) = \left\langle e^t, \frac{1}{3}t^3 + 2, \frac{1}{6}t^3 + \frac{1}{2}t^2 + t + 2\right\rangle$.

(b) i. Since the lines intersect, we know that there exist s, t such that

$$7 - 2t = 3 + 2s \quad (1)$$
$$t = -3 + 4s \quad (2)$$
$$-4 - t = -8 + 3s \quad (3)$$

Substituting (2) into (1) yields $7 - 2(-3 + 4s) = 3 + 2s \implies s = 1$. Eq. (2) then gives $t = 1$ and (3) is satisfied using these values. The lines intersect at $(5, 1, -5)$.

ii. A point in the plane is $(5, 1, -5)$. We need a normal vector, which is obtained as the cross product of the direction vectors of the two lines,

$$\langle -2, 1, -1\rangle \times \langle 2, 4, 3\rangle = \langle 7, 4, -10\rangle$$

Then $7(x - 5) + 4(y - 1) - 10(z + 5) = 0 \implies 7x + 4y - 10z = 89$

iii. The vector equation of the line is $\mathbf{r}(t) = \langle 5, 1, -5\rangle + t\langle 7, 4, -10\rangle$ giving the symmetric equations

$$\frac{x-5}{7} = \frac{y-1}{4} = \frac{z+5}{-10}$$

iv. The point where the line intersects the plane is found by substituting the parametric equations of the line into the plane's equation to find the value of t at the intersection point:

$$5 + 7t + 1 + 4t + -5 - 10t = 2 \implies t = 1$$

giving the point of intersection as $(12, 5, -15)$.

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