INSTRUCTIONS: Write your name and your instructor's name on the front of your work. Work all problems. Show your work clearly. Note that a correct answer with incorrect or no supporting work may receive no credit, while an incorrect answer with relevant work may receive partial credit. Simplify all work to receive full credit.

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1. (34 pts) The Calc 3 space cadets are back! During their travel back to Earth they accidentally travel counter-clockwise on the path of the left half of $x^2 + y^2 = 1$ connected by a straight line.

   (a) (20 pts) Directly compute $\oint_C yx^2dx - x^3dy$.

   (b) (14 pts) Use a Calc 3 theorem to compute part (a).

   Solution:

   (a) The straight line can be parameterized by $\vec{r}_1(t) = \langle 0, t\rangle$ with $t \in [-1, 1]$. Then,

   $$dx = 0dt \text{ and } dy = 1dt$$

   The line integral on this segment is then

   $$\begin{aligned}&\int_{-1}^{1} \left[(y)x^2x'(t) - x^3y'(t)\right] dt \\ &= \int_{-1}^{1} \left[(t)(0)^2(0) - (0)^3(1)\right] dt \\ &= 0\end{aligned}$$

   The circle can be parameterized by $\vec{r}_2(t) = \langle -\sin(t), c(t)\rangle$ with $t \in [0, \pi]$. Then,

   $$dx = -\cos(t)dt \text{ and } dy = -\sin(t)dt$$

   The line integral on this segment is then

   $$\begin{aligned}&\int_{0}^{\pi} \left[(y)x^2x'(t) - x^3y'(t)\right] dt \\ &= \int_{0}^{\pi} \left[((t))(-\sin(t))^2(-\cos(t)) - (-\sin(t))^3(-\sin(t))\right] dt \\ &= \int_{0}^{\pi} -\sin^2(t)\left[\cos^2(t) + \sin^2(t)\right] dt \\ &= \int_{0}^{\pi} \frac{1-\cos(2t)}{2} dt \\ &= -\left[\frac{t}{2} - \frac{\sin(2t)}{4}\right]_{t=0}^{t=\pi} \\ &= -\frac{\pi}{2}\end{aligned}$$

   The total is then $0 - \frac{\pi}{2} = -\frac{\pi}{2}$

(b) Note that $Q(x,y) = -x^3$ and $P(x,y) = yx^2$. Then using Green's Theorem

$$\oint_C yx^2 dx - x^3 dy = \iint_D \left[\frac{\partial}{\partial x}\left(-x^3\right) - \frac{\partial}{\partial y}\left(yx^2\right)\right] dA$$

Since the area is half a circle we will use polar coordinates instead

$$= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_0^1 \left[-3r^2\cos^2(\theta) - r^2\cos^2(\theta)\right] r dr d\theta$$

$$= -\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos^2(\theta) d\theta$$

$$= -\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{1+\cos(2\theta)}{2} d\theta$$

$$= -\left[\frac{\theta}{2} + \frac{\sin(2\theta)}{4}\right]_{\theta=\frac{\pi}{2}}^{\theta=\frac{3\pi}{2}}$$

$$= -\frac{\pi}{2}$$

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2. (34 pts) The space cadets have entered an asteroid field! To escape they'll need to calculate some line integrals. These parts are unrelated:

   (a) (24 pts) Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}$ is the conservative vector field $\langle 9x^2 - 3y^2x^2, 4 - 2yx^3\rangle$ and C is the path parameterized as $\vec{r}(t) = \langle 3 - t^2, 5 - t\rangle$ with $t \in [-2, 3]$ using the Fundamental Theorem for Line Integrals.

   (b) (10 pts) Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where C is the ellipse $\frac{(x-5)^2}{4} + \frac{y^2}{9} = 1$ counter-clockwise if $\int_L \vec{F} \cdot d\vec{r}$ is path independent (L being any path). Be sure to motivate your answer. Note: There isn't a listed vector field $\vec{F}$ on purpose.

Solution:

(a) First we need to find the scalar function $f(x,y)$ such that $\vec{F} = \nabla f$. Note,

$$\nabla f = \langle 9x^2 - 3y^2x^2, 4 - 2yx^3\rangle = \langle f_x, f_y\rangle$$

Then,

$$f_x = 9x^2 - 3y^2x^2$$
$$\Rightarrow$$
$$f(x,y) = 3x^2 - y^2x^3 + g(y)$$

where $g(y)$ is some function of y. Taking a y derivative,

$$f_y = -2yx^3 + g'(y) = 4 - 2yx^3$$
$$\Rightarrow$$
$$g'(y) = 4$$
$$\Rightarrow$$
$$g(y) = 4y + C$$

Then,

$$f(x,y) = 3x^3 - y^2x^3 + 4y + C$$

Note that $\vec{r}(-2) = \langle -1, 7\rangle$ and $\vec{r}(3) = \langle -6, 2\rangle$. Using the Fundamental Theorem for Line Integrals,

$$\int_C \vec{F} \cdot d\vec{r} = f(-6,2) - f(-1,7)$$
$$= 224 - 74 = 150$$

(b) Since we have path independence we know $\vec{F}$ is conservative ($\nabla \times \vec{F} = \vec{0}$). Since C is closed Stokes' Theorem then says

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = 0$$

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3. (32 pts) The space cadet's ship is receiving the final plans for re-entry into Earth's atmosphere, but a few more calculations need to done beforehand. These parts are unrelated:

   (a) (16 pts) Given a function $f(x, y)$ with continuous second order partial derivatives and $x = 2u + uv$ and $y = v^2 - u$, what is $\frac{\partial^2 f}{\partial v^2}$ (you may have to leave certain expressions general)?

   (b) (16 pts) Find the second order Taylor approximation of $f(x, y) = e^{-(x^2+y^2)}$ centered at the origin.

<u>**Solution:**</u>

(a) First by the chain rule

$$f_u = f_x x_v + f_y y_v = -u f_x + 2v f_y$$

For the second derivative

$$\begin{aligned} f_{uu} &= \left(\frac{\partial}{\partial v} u\right) f_x + u \left(\frac{\partial}{\partial v} f_x\right) + \left(\frac{\partial}{\partial v} 2v\right) f_y + 2v \left(\frac{\partial}{\partial v} f_y\right) \\ &= u \left(\frac{\partial}{\partial v} f_x\right) + 2 f_y + 2v \left(\frac{\partial}{\partial v} f_y\right) \end{aligned}$$

Well, f_{xv} and f_{yv} can both be calculated with the chain rule

$$\begin{aligned} \left(\frac{\partial}{\partial v} f_x\right) &= f_{xx} x_v + f_{xy} y_v \\ &= u f_{xx} + 2v f_{xy} \\ \left(\frac{\partial}{\partial v} f_y\right) &= f_{yx} x_v + f_{yy} y_v \\ &= u f_{yx} + 2v f_{yy} \end{aligned}$$

Thus,

$$\begin{aligned} f_{uu} &= u \left[u f_{xx} + 2v f_{xy}\right] + 2 f_y + 2v \left[u f_{yx} + 2v f_{yy}\right] \\ &= u^2 f_{xx} + 2 f_y + 4uv f_{yx} + 4v^2 f_{yy} \end{aligned}$$

(b) First

$$\begin{aligned} f &= e^{-(x^2+y^2)} \Rightarrow f(0,0) = 1 \\ f_x &= -2x e^{-(x^2+y^2)} \Rightarrow f_x(0,0) = 0 \\ f_y &= -2y e^{-(x^2+y^2)} \Rightarrow f_y(0,0) = 0 \\ f_{xx} &= (4x^2 - 2) e^{-(x^2+y^2)} \Rightarrow f_{xx}(0,0) = -2 \\ f_{yy} &= (4y^2 - 2) e^{-(x^2+y^2)} \Rightarrow f_{yy}(0,0) = -2 \\ f_{xy} &= 4xy e^{-(x^2+y^2)} \Rightarrow f_{xy}(0,0) = 0 \end{aligned}$$

Then the second order Taylor approximation centered at the origin is

$$\begin{aligned} f(x, y) &\approx f(0,0) + f_x(0,0)x + f_y(0,0)y + \frac{1}{2} f_{xx} x^2 + f_{xy}(0,0)xy + \frac{1}{2} f_{yy} y^2 \\ &= 1 - x^2 - y^2 \end{aligned}$$

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4. (20 pts) In preparation for the return to Earth's atmosphere the cadet's ship surrounds itself with a force field in the shape of the box $-1 \leq x \leq 2, 0 \leq y \leq 1$, and $1 \leq z \leq 4$. What is the flux of cosmic rays $\vec{F} = \langle \sin(\pi x), zy^3, z^2 + 4x \rangle$ out of the force field? Set up, but <u>DO NOT EVALUATE</u>, the calculation <u>without</u> directly computing any surface area integrals.

Solution:
Using Gauss's Theorem

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \nabla \cdot \vec{F} dV$$

First we calculate

$$\nabla \cdot \vec{F} = \pi \cos(\pi x) + 32y^2 + 2z$$

Then,

$$\iint_S \vec{F} \cdot d\vec{S} = \int_{-1}^{2} \int_{0}^{1} \int_{1}^{4} \left[\pi \cos(\pi x) + 32y^2 + 2z\right] dz dy dx$$

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5. (30 pts) To finally land the spaceship the cadets need to use Stokes' Theorem to evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \langle -yz, 4y + 1, xy \rangle$ and C is the circle of radius 3 perpendicular to the y-axis centered at $(0, 4, 0)$ with a clockwise rotation when looking down the y-axis from positive to negative.

Solution:
First we need that $\nabla \times \vec{F} = \langle x, -2y, z \rangle$. Next, a surface with the boundary of C is the plane $y = 4$ cut by C (making $g(x, y, z) = y$). The projection of this surface onto the xz-plane is the circle of radius 3 ($x^2 + z^2 = 9$).

$$\nabla g = \langle 0, 1, 0 \rangle$$
$$|\nabla g \cdot \hat{j}| = 1$$

The Right Hand Rule tells us the normal should be pointing in the negative y direction, thus we choose $\hat{n} = -\nabla g$.

$$\begin{aligned}(\nabla \times \vec{F}) \cdot d\vec{S} &= \langle x, -2y, z \rangle \cdot \langle 0, -1, 0 \rangle dA \\ &= 2y dA \\ &\text{We need to remove all } y\text{'s using } g(x, y, z) = y = 4 \\ &= 8 dA\end{aligned}$$

Thus, using Stokes' Theorem we have

$$\begin{aligned}\oint_C \vec{F} \cdot d\vec{r} &= \iint_S (\nabla \times \vec{F}) \cdot d\vec{S} \\ &= \int_0^3 \int_0^{2\pi} 8r d\theta dr \\ &= 8(\pi 3^2) \\ &= 72\pi\end{aligned}$$

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