1. (34 pts) The Calc 3 space cadets are back! During their travel back to Earth they accidentally travel counter-clockwise on the path of the left half of $x^2 + y^2 = 1$ connected by a straight line.

(a) (20 pts) Directly compute $\oint_C yx^2\,dx - x^3\,dy$.

(b) (14 pts) Use a Calc 3 theorem to compute part (a).

Solution:

(a) The straight line can be parameterized by $\vec{r}_1(t) = \langle 0, t \rangle$ with $t \in [-1, 1]$. Then,

$$dx = 0 \, dt \text{ and } dy = 1 \, dt$$

The line integral on this segment is then

$$\int_{-1}^{1} [(y)x^2 x'(t) - x^3 y'(t)] \, dt$$

$$= \int_{-1}^{1} [((0)^2)(0) - (0)^3(1)] \, dt$$

$$= 0$$

The circle can be parameterized by $\vec{r}_2(t) = \langle -\sin(t), c(t) \rangle$ with $t \in [0, \pi]$. Then,

$$dx = -\cos(t) \, dt \text{ and } dy = -\sin(t) \, dt$$

The line integral on this segment is then

$$\int_{0}^{\pi} [(y)x^2 x'(t) - x^3 y'(t)] \, dt$$

$$= \int_{0}^{\pi} [((t))(- \sin(t))^2(- \cos(t)) - (- \sin(t))^3(- \sin(t))] \, dt$$

$$= \int_{0}^{\pi} - \sin^2(t) [\cos^2(t) + \sin^2(t)] \, dt$$

$$= \int_{0}^{\pi} 1 - \cos(2t) \, dt$$

$$= \left[ \frac{t - \sin(2t)}{2} \right]_{t=0}^{t=\pi}$$

$$= \frac{\pi}{2}$$

The total is then $0 - \frac{\pi}{2} = -\frac{\pi}{2}$. 

(b) Note that \(Q(x, y) = -x^3\) and \(P(x, y) = yx^2\). Then using Green’s Theorem
\[
\oint_C yx^2 \, dx - x^3 \, dy = \iint_D \left[ \frac{\partial}{\partial x} (-x^3) - \frac{\partial}{\partial y} (yx^2) \right] \, dA
\]
Since the area is half a circle we will use polar coordinates instead
\[
= \int_0^{\frac{\pi}{2}} \int_0^1 \left[ -3r^2 \cos^2(\theta) - r^2 \cos^2(\theta) \right] \, r \, dr \, d\theta
\]
\[
= -\int_0^{\frac{\pi}{2}} \cos^2(\theta) \, d\theta
\]
\[
= -\int_0^{\frac{\pi}{2}} \frac{1 + \cos(2\theta)}{2} \, d\theta
\]
\[
= -\left[ \frac{\theta}{2} + \frac{\sin(2\theta)}{4} \right]_{\theta=\frac{\pi}{2}}^{\theta=\frac{\pi}{2}}
\]
\[
= -\frac{\pi}{2}
\]
\[\square\]

2. (34 pts) The space cadets have entered an asteroid field! To escape they’ll need to calculate some line integrals. These parts are unrelated:

(a) (24 pts) Evaluate \(\int_C \vec{F} \cdot d\vec{r}\) where \(\vec{F}\) is the conservative vector field \((9x^2 - 3y^2x^2, 4 - 2yx^3)\) and \(C\) is the path parameterized as \(\vec{r}(t) = (3 - t^2, 5 - t)\) with \(t \in [-2, 3]\) using the Fundamental Theorem for Line Integrals.

(b) (10 pts) Evaluate \(\int_C \vec{F} \cdot d\vec{r}\) where \(C\) is the ellipse \(\frac{(x-5)^2}{4} + \frac{y^2}{9} = 1\) counter-clockwise if \(\int_L \vec{F} \cdot d\vec{r}\) is path independent \((L\) being any path). Be sure to motivate your answer. Note: There isn’t a listed vector field \(\vec{F}\) on purpose.

Solution:

(a) First we need to find the scalar function \(f(x, y)\) such that \(\vec{F} = \nabla f\). Note,
\[
\nabla f = (9x^2 - 3y^2x^2, 4 - 2yx^3) = (f_x, f_y)
\]
Then,
\[
f_x = 9x^2 - 3y^2x^2
\]
\[
\Rightarrow
\]
\[
f(x, y) = 3x^2 - y^2x^3 + g(y)
\]
where \(g(y)\) is some function of \(y\). Taking a \(y\) derivative,
\[
f_y = -2yx^3 + g'(y) = 4 - 2yx^3
\]
\[
\Rightarrow
\]
\[
g'(y) = 4
\]
\[
\Rightarrow
\]
\[
g(y) = 4y + C
\]
Then,
\[
f(x, y) = 3x^3 - y^2x^3 + 4y + C
\]

Note that \(\vec{r}(-2) = (-1, 7)\) and \(\vec{r}(3) = (-6, 2)\). Using the Fundamental Theorem for Line Integrals,
\[
\int_C \vec{F} \cdot d\vec{r} = f(-6, 2) - f(-1, 7)
\]
\[
= 224 - 74 = 150
\]

(b) Since we have path independence we know \(\vec{F}\) is conservative \((\nabla \times \vec{F} = \vec{0})\). Since \(C\) is closed Stokes’ Theorem then says
\[
\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = 0
\]
3. (32 pts) The space cadet’s ship is receiving the final plans for re-entry into Earth’s atmosphere, but a few more calculations need to done beforehand. These parts are unrelated:

(a) (16 pts) Given a function \( f(x, y) \) with continuous second order partial derivatives and \( x = 2u + uv \) and \( y = v^2 - u \), what is \( \frac{\partial^2 f}{\partial u^2} \) (you may have to leave certain expressions general)?

(b) (16 pts) Find the second order Taylor approximation of \( f(x, y) = e^{-(x^2 + y^2)} \) centered at the origin.

Solution:

(a) First by the chain rule

\[
f_u = f_x v + f_y v = -uf_x + 2v f_y
\]

For the second derivative

\[
f_{uu} = \left( \frac{\partial}{\partial u} \right)^2 f_u = \left( \frac{\partial}{\partial u} \right) \left( \frac{\partial}{\partial v} f_u \right) = u \left( \frac{\partial}{\partial v} f_x \right) + 2f_y + 2v \left( \frac{\partial}{\partial v} f_y \right)
\]

Well, \( f_{xv} \) and \( f_{yv} \) can both be calculated with the chain rule

\[
\left( \frac{\partial}{\partial u} f_x \right) = f_{xx} v + f_{xy} v^2 = uf_{xx} + 2vf_{xy}
\]

\[
\left( \frac{\partial}{\partial u} f_y \right) = f_{yx} v + f_{yy} v^2 = uf_{yx} + 2vf_{yy}
\]

Thus,

\[
f_{uu} = u \left[ uf_{xx} + 2vf_{xy} \right] + 2f_y + 2v \left[ uf_{yx} + 2vf_{yy} \right]
\]

\[
= u^2 f_{xx} + 2f_y + 4uvf_{yx} + 4v^2 f_{yy}
\]

(b) First

\[
f = e^{-(x^2 + y^2)} \Rightarrow f(0, 0) = 1
\]

\[
f_x = -2xe^{-(x^2 + y^2)} \Rightarrow f_x(0, 0) = 0
\]

\[
f_y = -2ye^{-(x^2 + y^2)} \Rightarrow f_y(0, 0) = 0
\]

\[
f_{xx} = (4x^2 - 2)e^{-(x^2 + y^2)} \Rightarrow f_{xx}(0, 0) = -2
\]

\[
f_{yy} = (4y^2 - 2)e^{-(x^2 + y^2)} \Rightarrow f_{yy}(0, 0) = -2
\]

\[
f_{xy} = 4xye^{-(x^2 + y^2)} \Rightarrow f_{xy}(0, 0) = 0
\]

Then the second order Taylor approximation centered at the origin is

\[
f(x, y) \approx f(0, 0) + f_x(0, 0)x + f_y(0, 0)y + \frac{1}{2} f_{xx}(0, 0) x^2 + f_{xy}(0, 0) xy + \frac{1}{2} f_{yy}(0, 0) y^2
\]

\[
= 1 - x^2 - y^2
\]

4. (20 pts) In preparation for the return to Earth’s atmosphere the cadet’s ship surrounds itself with a force field in the shape of the box \(-1 \leq x \leq 2, 0 \leq y \leq 1, \text{ and } 1 \leq z \leq 4\). What is the flux of cosmic rays \( \vec{F} = (\sin(\pi x), zy^3, z^2 + 4x) \) out of the force field? Set up, but DO NOT EVALUATE, the calculation without directly computing any surface area integrals.
**Solution:**

Using Gauss’s Theorem

\[
\iiint_S \vec{F} \cdot d\vec{S} = \iiint_E \nabla \cdot \vec{F} dV
\]

First we calculate

\[\nabla \cdot \vec{F} = \pi \cos(\pi x) + 32y^2 + 2z\]

Then,

\[
\iiint_S \vec{F} \cdot d\vec{S} = \int_{-1}^{2} \int_{0}^{1} \int_{1}^{4} \left[ \pi \cos(\pi x) + 32y^2 + 2z \right] dzdydx
\]

5. (30 pts) To finally land the spaceship the cadets need to use Stokes’ Theorem to evaluate \[\oint_C \vec{F} \cdot d\vec{r}\] where \[\vec{F} = (-y, 4y + 1, xy)\] and \(C\) is the circle of radius 3 perpendicular to the y-axis centered at \((0, 4, 0)\) with a clockwise rotation when looking down the y-axis from positive to negative.

**Solution:**

First we need that \(\nabla \times \vec{F} = (x, -2y, z)\). Next, a surface with the boundary of \(C\) is the plane \(y = 4\) cut by \(C\) (making \(g(x, y, z) = y\)). The projection of this surface onto the \(xz\)-plane is the circle of radius 3 \((x^2 + z^2 = 9)\).

\[
\nabla g = (0, 1, 0)
\]

\[
|\nabla g \cdot \hat{j}| = 1
\]

The Right Hand Rule tells us the normal should be pointing in the negative y direction, thus we choose \(\hat{n} = -\nabla g\).

\[
(\nabla \times \vec{F}) \cdot d\vec{S} = (x, -2y, z) \cdot (0, -1, 0) dA = 2ydA
\]

We need to remove all \(y\)’s using \(g(x, y, z) = y = 4\)

\[= 8dA\]

Thus, using Stokes’ Theorem we have

\[
\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}
\]

\[= \int_{0}^{3} \int_{0}^{2\pi} 8r \, d\theta \, dr\]

\[= 8(\pi 3^2)\]

\[= 72\pi\]