INSTRUCTIONS: Write your name and your instructor's name on the front of your work. Work all problems. Show your work clearly. Note that a correct answer with incorrect or no supporting work may receive no credit, while an incorrect answer with relevant work may receive partial credit. Simplify all work to receive full credit.

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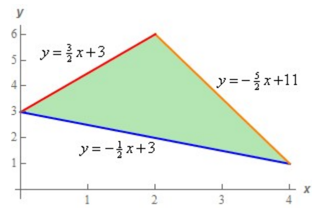
1. (25 pts) Dr. Strang (a real mathematician by the way) is helping by creating a math-mystical portal in the shape of a triangle with vertices $(0,3)$, $(4,1)$, and $(2,6)$ in xy-space (let's call this region R). In order to complete the spell he needs to transform $\iint_R (x+2y)dA$ using $x = \frac{1}{2}(u-v)$ and $y = \frac{1}{4}(3u+v+12)$. Set up, but DO NOT EVALUATE, the resulting new integral in uv-space.

Solution:

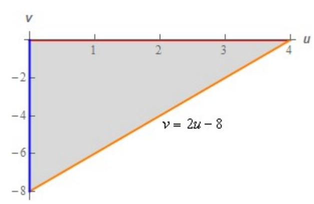
The three points allow us to find the equations of each side of the triangle:



$$y = \frac{3}{2}x + 3$$
$$y = -\frac{5}{2}x + 11$$
$$y = -\frac{1}{2}x + 3$$

If substitute $x = \frac{1}{2}(u-v)$ and $\frac{1}{4}(3u+v+12)$ into these three equations:

$$\frac{1}{4}(3u+v+12) = \frac{3}{4}(u-v) + 3 \Rightarrow v = 0$$
$$\frac{1}{4}(3u+v+12) = -\frac{5}{4}(u-v) + 11 \Rightarrow v = 2u - 8$$
$$\frac{1}{4}(3u+v+12) = -\frac{1}{4}(u-v) + 3 \Rightarrow u = 0$$



This forms a right triangle in the uv-plane. Next we find the Jacobian,

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} \end{vmatrix} = \frac{1}{2}$$

The integral in uv-space is then,

$$\int_0^4 \int_{2u-8}^0 \frac{1}{2}\left[\frac{1}{2}(u-v) + 2\left(\frac{1}{4}\right)(3u+v+12)\right] dvdu$$
$$= \int_0^4 \int_{2u-8}^0 (u+3)dvdu$$

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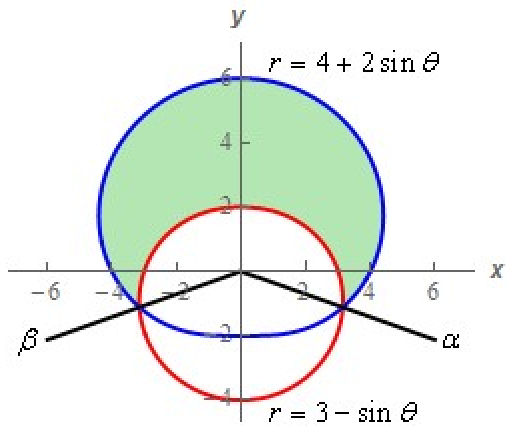
2. (20 pts) Captain Mathematical is focusing her energy attacks in the area inside $r = 4+2\sin(\theta)$ and outside $r = 3-\sin(\theta)$. Sketch the region and set up, but DO NOT EVALUATE, the integral that represents this area.

Solution:

The region looks like:



First we set these two curves equal to find intersection points.

$$\begin{aligned} 4+2\sin(\theta) &= 3-\sin(\theta) \\ 3\sin(\theta) &= -1 \\ \sin(\theta) &= -\frac{1}{3} \end{aligned}$$

Note that $\arcsin\left(-\frac{1}{3}\right)$ only refers to α due to arcsine only being defined on $[-\pi,\pi]$, but due to symmetry of this shape we can find that $\pi-\arcsin\left(-\frac{1}{3}\right)$. Plugging these angles into both functions ensures that these are in fact the points of intersection. Also, plugging in any angle in between (like $\frac{\pi}{2}$) ensures these curves are tracing the same region at the same values of θ. Thus, our area integral is,

$$\int_{\arcsin\left(-\frac{1}{3}\right)}^{\pi-\arcsin\left(-\frac{1}{3}\right)} \int_{3-\sin(\theta)}^{4+2\sin(\theta)} r\,dr\,d\theta$$

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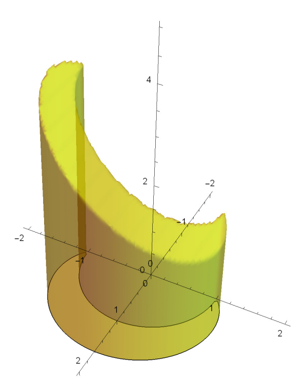
3. (30 pts) Iron-math is flying around the battlefield. He spots groups of enemies sneaking behind two structures. In order to decide how much power to use in his attack he first needs data on these structures. Set up, but DO NOT EVALUATE the following integrals calculating:

   (a) The mass for the region bounded by $z=0$, outside of $r=1$, inside of $r=1+\cos(\theta)$, and $z=3-y$ with density $D(x,y,z)=x^2+y^2+z^2$ in the order $dz\,dr\,d\theta$.

   (b) The volume for the region bounded below by the xy-plane, on the sides by a sphere of radius 2 centered at the origin, and on the top by the cone $3z^2=x^2+y^2$ in the order $d\rho\,d\phi\,d\theta$

Solution:

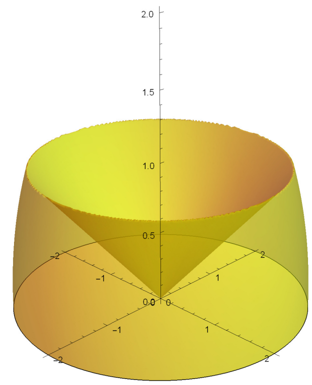
(a) The region looks like



The z-direction is bounded by $z = 0$ and $z = 3 - y = 3 - r\sin(\theta)$. The r-direction is bounded by $r = 1$ and $r = 1 + \cos(\theta)$. The θ bounds can be found by setting the r bounds equal to find the angle of intersection, leading to $\theta = -\frac{\pi}{2}$ and $\theta = \frac{\pi}{2}$ (being careful to which is the lower/upper bound). The density function in cylindrical coordinates is $D(r, \theta, z) = r^z + z^2$. The mass integral is then

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{1}^{1+\cos(\theta)} \int_{0}^{3-r\sin(\theta)} (z^2 + r^2) r dz dr d\theta$$

(b) The region looks like



Start by converting $2z^2 = x^2 + y^2$ in spherical coordinates.

$$3\rho^2\cos^2(\phi) = \rho^2\sin^2(\phi)^2\cos^2(\theta) + \rho^2\sin^2(\phi)^2\sin^2(\theta)$$
$$\Rightarrow$$
$$3\cos^2(\phi) = \sin^2(\phi)$$
$$\Rightarrow$$
$$\sqrt{3} = \tan(\phi)$$

This means $\phi = \frac{\pi}{3}$. This gives our lower bound for ϕ. The sphere of radius 2 gives us that the ρ-direction is bounded by $\rho = 0$ and $\rho = 2$. The volume is then

$$\int_{0}^{2\pi} \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \int_{0}^{2} \rho^2 \sin(\phi) d\rho d\phi d\theta$$

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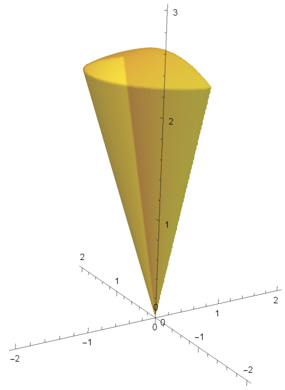
4. (25 pts) Black Widom (Widom is also a real mathematician) is attempting to steal the villain's secret plans. She needs to break the computer's encryption first. In order to do so she needs to rewrite the integral

$$\int_{-1}^{0} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{6x^2+6y^2}}^{\sqrt{7-x^2-y^2}} 18y dz dy dx$$

in a different coordinate system in order to evaluate it. This time set up AND evaluate this integral. **Hint:** If you are struggling evaluating the integral maybe try a different integration order.

Solution:
First, the z-bounds go from $z = \sqrt{6x^2 + 6y^2}$ (a cone) $z = \sqrt{7 - x^2 - y^2}$ (the upper portion of a sphere of radius $\sqrt{7}$ centered at the origin). The projection of this object onto the xy-plane is bounded by $-1 \leq x \leq 0$ and $-\sqrt{1 - x^2} \leq y \leq \sqrt{1 - x^2}$, or the left half of a circle of radius 1 centered at the origin. The object looks like



The equation of the cone $z = \sqrt{6x^2 + 6y^2}$ tells us the angle $\phi = \arctan\left(\frac{1}{\sqrt{6}}\right)$ via

$$\rho\cos(\phi) = \sqrt{6\left(\rho^2\sin^2(\phi)\cos^2(\theta) + \rho^2\sin^2(\phi)\sin^2(\theta)\right)} = \sqrt{6}\rho\sin(\phi)$$
$$\Rightarrow$$
$$\tan(\phi) = \frac{1}{\sqrt{6}}$$

The θ bounds need to form the left half of the circle in the xy-plane, meaning $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$. The converted integral is then

$$\int_0^{\arctan\left(\frac{1}{\sqrt{6}}\right)} \int_0^{\sqrt{7}} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(18\rho\sin(\phi)\sin(\theta)\right)\left(\rho^2\sin(\phi)\right) d\theta d\rho d\phi$$
$$= \int_0^{\arctan\left(\frac{1}{\sqrt{6}}\right)} \int_0^{\sqrt{7}} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 18\rho^3\sin^2(\phi)\sin(\theta) d\theta d\rho d\phi$$
$$= \int_0^{\arctan\left(\frac{1}{\sqrt{6}}\right)} \int_0^{\sqrt{7}} \left[\rho^3\sin^2(\phi)\cos(\theta)\right]_{\theta=\frac{\pi}{2}}^{\theta=\frac{3\pi}{2}} d\rho d\phi$$
$$= \int_0^{\arctan\left(\frac{1}{\sqrt{6}}\right)} \int_0^{\sqrt{7}} 0 d\rho d\phi$$
$$= 0$$

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