INSTRUCTIONS: Write your name and your instructor's name on the front of your work. Work all problems. Show your work clearly. Note that a correct answer with incorrect or no supporting work may receive no credit, while an incorrect answer with relevant work may receive partial credit.

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1. (20 points) Prince Leo has begun a math-mystical journey across the realm. He comes to long stone bridge over a deadly canyon. A strange man in grey clothing yells across "You shall not pass... without solving these riddles". The man asks to consider a function of two variable, $f(x, y)$, that has continuous second order derivatives. He states that $x = u^2 + 3v$ and $y = uv$ and asks Leo to find the following (you may have to leave certain expressions general):

   (a) $\frac{\partial f}{\partial u}$ (b) $\frac{\partial}{\partial u}\frac{\partial f}{\partial x}$ (c) $\frac{\partial}{\partial u}\frac{\partial f}{\partial y}$ (d) $\frac{\partial^2 f}{\partial u^2}$

   <u>Solution:</u>

   (a) By the chain rule,

   $$f_u = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = 2uf_x + vf_y$$

   (b) By the chain rule,

   $$\begin{aligned}\frac{\partial}{\partial u}\frac{\partial f}{\partial x} &= \frac{\partial x}{\partial u}\left(\frac{\partial}{\partial x}f_x\right) + \frac{\partial y}{\partial u}\left(\frac{\partial}{\partial y}f_x\right)\\ &= 2uf_{xx} + vf_{yx}\end{aligned}$$

   (c) By the chain rule,

   $$\begin{aligned}\frac{\partial}{\partial u}\frac{\partial f}{\partial y} &= \frac{\partial x}{\partial u}\left(\frac{\partial}{\partial x}f_y\right) + \frac{\partial y}{\partial u}\left(\frac{\partial}{\partial y}f_y\right)\\ &= 2uf_{xy} + vf_{yy}\end{aligned}$$

   (d)

   $$\begin{aligned}\frac{\partial^2 f}{\partial u^2} &= \frac{\partial}{\partial u}f_u = \frac{\partial}{\partial u}\left[2u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y}\right]\\ &= 2\frac{\partial f}{\partial x} + 2u\frac{\partial}{\partial u}\frac{\partial f}{\partial x} + v\frac{\partial}{\partial u}\frac{\partial f}{\partial y}\\ &\text{By substituting part b and c}\\ &= 2f_x + 4u^2 f_{xx} + 4uvf_{xy} + v^2 f_{yy}\end{aligned}$$

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2. (30 points) After crossing the bridge Leo enters a large cavern. Inside he finds magic light swirling around a large sphere (centered at the origin) of radius 6m. Leo finds that the amount of light can be described by $f(x, y, z) = y^2 - 10z$ (allowing magic light to have negative values). What is the maximum and minimum values of magic light on the surface of the sphere?

   <u>Solution:</u>
   First, the function to maximize/minimize is $f(x, y, z) = y^2 - 10z$ with a constraint function of $g(x, y, z) = x^2 + y^2 + z^2 = 36$.

   $$\begin{aligned}f_x &= \lambda g_x \Rightarrow 0 = \lambda x\\ f_y &= \lambda g_y \Rightarrow y = \lambda y\\ f_z &= \lambda g_z \Rightarrow -5 = \lambda z\\ x^2 &+ y^2 + z^2 = 36\end{aligned}$$

   Since $-5 = \lambda z$ we know $z \neq 0$ or $\lambda \neq 0$. This means that $x = 0$ from the first equation. The second equation tells us that either $y = 0$ or $\lambda = 1$.
   Case: $y = 0$

   $$\text{If } y = 0 \Rightarrow z^2 = 36 \Rightarrow z = \pm 6$$
   $$\text{This give the points: } (0, 0, -6),\ (0, 0, 6)$$

Case: $\lambda = 1$

$$\text{If } \lambda = 1 \Rightarrow z = -5$$
$$\Rightarrow y^2 + 25 = 36 \Rightarrow y^2 = 11 \Rightarrow y = \pm\sqrt{11}$$
$$\text{This give the points: } (0, -\sqrt{11}, -5),\ (0, \sqrt{11}, -5)$$

Testing these points:

$$f(0, 0, -6) = 60$$
$$f(0, 0, 6) = -60$$
$$f(0, -\sqrt{11}, -5) = 61$$
$$f(0, \sqrt{11}, -5) = 61$$

Thus, the maximum is 61 and the minimum is -60. ■

3. (30 points) Leo exits the magic cavern near the top of Mount Newton. Locally, The elevation is described by $h(x, y) = 7x - 8y + 2xy - x^2 + y^3$.
   (a) Find and classify all the critical points of Mount Newton.
   (b) Leo is currently at $(1, -1, 11)$ when a mighty dragon descends from the sky breathing fire. Leo quickly runs down the mountain in the steepest direction. Find a full 3-dimensional vector that points in the direction Leo begins running.

Solution:

(a) First to find the critical points:

$$h_x = 7 + 2y - 2x$$
$$h_y = -8 + 2x + 3y^2$$
$$\Rightarrow$$
$$7 + 2y - 2x = 0$$
$$-8 + 2x + 3y^2 = 0$$

From equation 2 we have $x = 4 - \frac{3}{2}y^2$. Plugging this into equation 1,

$$0 = 7 + 2y - 2\left(4 - \frac{3}{2}y^2\right)$$
$$= 3y^2 + 2y - 1$$
$$= (3y - 1)(y + 1)$$
$$\Rightarrow$$
$$y = -1, \frac{1}{3}$$

This gives us the points $\left(\frac{5}{2}, -1\right)$ and $\left(\frac{23}{6}, \frac{1}{3}\right)$. To classify these critical points we can use the second derivative test.

$$D(x, y) = h_{xx}h_{yy} - [h_{xy}]^2$$
$$= (-2)(6y) - (2)^2$$
$$= -12y - 4$$
$$\text{Plugging in our points:}$$
$$D\left(\frac{5}{2}, -1\right) = 8$$
$$D\left(\frac{23}{6}, \frac{1}{3}\right) = -8$$

$\left(\frac{23}{6}, \frac{1}{3}\right)$ is a saddle point since $D < 0$ and $\left(\frac{5}{2}, -1\right)$ is a local maximum since $D > 0$ and $h_{xx} < 0$.

(b) In the xy-plane this direction is $-\nabla h(1,-1) = \langle -3, 3\rangle$. The directional derivative in this direction is $-||\nabla h(1,-1)|| = -\sqrt{18}$. Thus, a vector in the direction Leo is running will point down $-\sqrt{18}$ for every 1 unit in the xy-plane. Well, $||\langle -3, 3\rangle|| = \sqrt{18}$ meaning the vector pointing in the direction that Leo begins running in is $\langle -3, 3, -18\rangle$ (or any scalar multiple of this vector).

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4. (20 points) Leo continues running from the dragon until he encounters a sudden cliff. With nowhere to run he turns to fight the beast. Surprisingly, the dragon challenges Leo to a math duel. The dragons roars to

(a) Find the second order Taylor approximation to $e^x \sin(y)$ at the origin.

(b) Estimate the error in this approximation if $|x| \leq 0.1$ and $|y| \leq 0.1$

Solution:

(a) First we need to calculate the required values and derivatives:

$$\begin{aligned} f(x,y) = e^x \sin(y) &\Rightarrow f(0,0) = 0 \\ f_x(x,y) = e^x \sin(y) &\Rightarrow f_x(0,0) = 0 \\ f_y(x,y) = e^x \cos(y) &\Rightarrow f_y(0,0) = 1 \\ f_{xx}(x,y) = e^x \sin(y) &\Rightarrow f_{xx}(0,0) = 0 \\ f_{xy}(x,y) = e^x \cos(y) &\Rightarrow f_{xy}(0,0) = 1 \\ f_{yy}(x,y) = -e^x \sin(y) &\Rightarrow f_{xx}(0,0) = 0 \end{aligned}$$

Thus, the second order Taylor approximation is

$$\begin{aligned} T(x,y) &= f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) + \frac{1}{2} f_{xx}(0,0)(x-0)^2 + f_{xy}(0,0)(x-0)(y-0) + \frac{1}{2}(y-0)^2 \\ &= y + xy \end{aligned}$$

(b) To estimate the error bound on this approximation we require the third derivatives

$$\begin{aligned} f_{xxx}(x,y) &= e^x \sin(y) \\ f_{xxy}(x,y) &= e^x \cos(y) \\ f_{xyy}(x,y) &= -e^x \sin(y) \\ f_{yyy}(x,y) &= -e^x \cos(y) \end{aligned}$$

These are all bounded by e^x and since $|x| \leq \frac{1}{10}$ we have they are all bounded by $e^{\frac{1}{10}}$. The error bound is then,

$$\begin{aligned} |E(x,y)| &\leq \frac{1}{3!}\left[|f_{xxx}||x|^3 + 3|f_{xxy}||x|^2|y| + 3|f_{xyy}||x||y|^2 + |f_{yyy}||y|^3\right] \\ &= \frac{1}{6}\left[e^{\frac{1}{10}}\left(\frac{1}{10^3} + \frac{3}{10^3} + \frac{3}{10^3} + \frac{1}{10^3}\right)\right] \\ &= \frac{8e^{\frac{1}{10}}}{6 \cdot 10^3} = \frac{4e^{\frac{1}{10}}}{3000} \end{aligned}$$

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After Leo returns from his journey he is finally ready to take the throne to become King Leonhard Euler.