INSTRUCTIONS: Write your name and your instructor's name on the front of your work. Work all problems. Show your work clearly. Note that a correct answer with incorrect or no supporting work may receive no credit, while an incorrect answer with relevant work may receive partial credit.

1. (20 points) The powerful math wizard, Algebrus Dumblefour, is creating mystical curves and surfaces for his class. He begins by creating a plane that includes the points $A(1,0,1)$, $B(2,2,1)$, and $C(1,2,2)$. He then creates the line $\frac{1-x}{2} = 1 - y = \frac{z-2}{2}$.

   (a) Find the equation of the plane.

   (b) Find the angle the line hits the plane.

   Solution:

   (a) First note:

$$\vec{AB} = \langle 2,2,1\rangle - \langle 1,0,1\rangle = \langle 1,2,0\rangle$$
$$\vec{AC} = \langle 1,2,2\rangle - \langle 1,0,1\rangle = \langle 0,2,1\rangle$$

   The cross product of these two vectors will give the normal direction,

$$\mathbf{n} = \vec{AB} \times \vec{AC} = \langle 2,-1,2\rangle$$

   Thus, the plane is given by

$$2(x-1) - y + 2(z-1) = 0 \Rightarrow 2x - y + 2z = 4$$

   (b) The angle the line hits the plane can be found be first finding the angle of the line to the normal of the plane. The direction of the line is $\mathbf{v} = \langle -2,-1,2\rangle$. The angle between the two vectors is then

$$\begin{aligned}\theta &= \arccos\left(\frac{\mathbf{n}\cdot\mathbf{v}}{||\mathbf{n}||\cdot||\mathbf{v}||}\right)\\ &= \arccos\left(\frac{-4+1+4}{3\cdot 3}\right)\\ &= \arccos\left(\frac{1}{9}\right)\end{aligned}$$

   The angle the line makes with the plane is then $\frac{\pi}{2} - \theta = \frac{\pi}{2} - \arccos\left(\frac{1}{9}\right)$

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2. (30 points) A Calc 3 space cadet is traveling along the path

$$\mathbf{C}(t) = \langle t^2, 9+t, \sqrt{6}t^{\frac{3}{2}}\rangle, \quad t \geq 0$$

   while Darth Mathious pursues along the path

$$\mathbf{D}(t) = \left\langle \frac{t^3}{3}, 4t, \sqrt{2}t^2 \right\rangle, \quad t \geq 0.$$

   (a) Where, if at all, would Darth Mathious intercept the space cadet?

   (b) Darth Mathious only has enough fuel to travel $\frac{32}{3}$ space units. Where is his ship when he runs out of fuel? Does he have enough fuel to capture the space cadet (provided his path allows him)? *Hint:* If you are having trouble solving for the time, try plugging in some integers to get you started and note $t^3 + 12t - 32 = 0$ only has one real root.

   Solution:

(a) To find any time these paths intersect we set

$$t^2 = \frac{t^3}{3} \Rightarrow t = 0, 3$$
$$9 + t = 4t \Rightarrow t = 3$$
$$\sqrt{6}t^{\frac{3}{2}} = \sqrt{2}t^2 \Rightarrow t = 0, 3.$$

Thus, at $t = 3$ Darth Mathious intersects the space cadet at the point $(9, 12, 9\sqrt{2})$.

(b) First start with $\mathbf{D}'(t) = \langle t^2, 4, 2\sqrt{2}t\rangle$. Then, note

$$||\mathbf{D}'(t)|| = \sqrt{t^4 + 16 + 8t^2} = \sqrt{(t^2+4)^2}$$
$$= t^2 + 4$$

Thus, the total distance traveled as a function of t is,

$$s(t) = \int_0^t ||\mathbf{D}'(u)||du = \int_0^t (u^2 + 4)du$$
$$= \frac{t^3}{3} + 4t$$

The only real valued solution to $\frac{t^3+12t}{3} = \frac{32}{3}$ is $t = 2$. This puts Darth Mathious at $\left(\frac{8}{3}, 8, 4\sqrt{2}\right)$. Since he would catch the space cadet at $t = 3$, he does **not** have enough fuel to capture the cadet.

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3. (30 points) Mathy McFly is in his time-traveling flying car trying to get back to the future. He follows the path

$$\mathbf{r}(t) = \langle \cos(2t), -\sin(2t), 4t\rangle, \quad t \geq 0$$

to avoid some flying debris. At time $t = \pi$ seconds a straight path opens up in the direction he is headed. He maintains his speed for another π seconds before he time-travels.

(a) What is the curvature, $\kappa(t)$, of the path with $0 \leq t \leq \pi$?

(b) At what time(s), if any, with $0 \leq t \leq \pi$ is the unit normal vector of the path parallel to the line parameterized by $\mathbf{L}(t) = \langle 1 + \sqrt{2}t, 2, 2\rangle$?

(c) Where is Mathy when he time-travels?

Solution:

(a) First start with $\mathbf{r}'(t) = \langle -2\sin(2t), -2\cos(2t), 4\rangle$. Then

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||} = \left\langle -\frac{\sin(2t)}{\sqrt{5}}, -\frac{\cos(2t)}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right\rangle.$$

We then need the magnitude of the derivative of the unit tangent vector,

$$||\mathbf{T}'(t)|| = \left|\left|\left\langle -\frac{2\cos(2t)}{\sqrt{5}}, \frac{2\sin(2t)}{\sqrt{5}}, 0\right\rangle\right|\right|$$
$$= \sqrt{\frac{4}{5}\cos^2(2t) + \frac{4}{5}\sin^2(2t)} = \frac{2}{\sqrt{5}}$$

The curvature is then,

$$\kappa(t) = \frac{||\mathbf{T}'(t)||}{||\mathbf{r}'(t)||} = \frac{\frac{2}{\sqrt{5}}}{2\sqrt{5}} = \frac{1}{5}$$

(b) The normal vector points in the direction of $\mathbf{T}'(t) = \left\langle -\frac{2\cos(2t)}{\sqrt{5}}, \frac{2\sin(2t)}{\sqrt{5}}, 0\right\rangle$. The line $\mathbf{L}(\mathbf{t})$ points in the direction $\langle\sqrt{2}, 0, 0\rangle$. These vectors are parallel when they are scalar multiples of each other. This only happens when $\mathbf{T}'(\mathbf{t}) = \langle a, 0, 0\rangle$ with $a \neq 0$. This can happen at $t = 0, \frac{\pi}{2}, \pi$.

(c) At $t = \pi$ Mathy has position given by $\mathbf{r}(\pi) = \langle 1, 0, 4\pi\rangle$ and a velocity given by $\mathbf{r}'(\pi) = \langle 0, -2, 4\rangle$. Thus, Mathy is located at

$$\langle 1, 0, 4\pi\rangle + \pi\langle 0, -2, 4\rangle = \langle 1, -2\pi, 8\pi\rangle$$

when he time-travels.

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4. (20 points) Mathy McFly has arrived to the future at the (unrelated from problem 3) point $\mathbf{r}(0) = \langle 0, 0, 10\rangle$. His on board navigation system logs his velocity as

$$\mathbf{v}(t) = \langle 2, 2t, t^2\rangle, \quad t \geq 0.$$

(a) What is the vector function for position?

(b) What are the tangential and normal components of acceleration?

Solution:

(a) The vector function of position can be found by

$$\begin{aligned}
\mathbf{r}(t) &= \mathbf{r}(0) + \int_0^t \mathbf{v}(u)du \\
&= \langle 0, 0, 10\rangle + \int_0^t \langle 2, 2u, u^2\rangle du \\
&= \langle 0, 0, 10\rangle + \left\langle 2t, t^2, \frac{t^3}{3}\right\rangle \\
&= \left\langle 2t, t^2, \frac{t^3}{3} + 10\right\rangle
\end{aligned}$$

(b) Start with acceleration $\mathbf{a}(t) = \langle 0, 2, 2t\rangle$. Next,

$$\begin{aligned}
a_T(t) &= \frac{d}{dt}||\mathbf{v}(t)|| \\
&= \frac{d}{dt}\left[\sqrt{4 + 4t^2 + t^4}\right] \\
&= \frac{d}{dt}\sqrt{(t^2 + 2)^2} \\
&= \frac{d}{dt}(t^2 + 2) \\
&= 2t
\end{aligned}$$

To find $a_N(t)$ note

$$\begin{aligned}
a_N(t) &= \sqrt{||\mathbf{a}(t)||^2 - a_T(t)^2} \\
&= \sqrt{(4 + 4t^2) - (4t^2)} = 2
\end{aligned}$$

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