NOTE: Any integrals that need to be evaluated will require integration techniques no more complicated than u-substitution.

1. [45 pts] A whale is swimming around the ocean collecting plankton in its baleen (like a filter). It swims in the plane $z = -1$. Beginning at the point $(x, y, z) = (0, -3, -1)$, it follows a quarter of the circle of radius 3 centered at $(0, 0, -1)$ with $y \le 0$ and $x \ge 0$ to the point $(x, y, z) = (3, 0, -1)$. From there it then swims along a straight line to the point where it started. The density of plankton, the food the whale eats, is given by $\delta(x, y, z) = xy^2(z + 2)$ plankton per meter.

   (a) [6 pts] Sketch the whale's path in the plane $z = -1$, making certain to show the correct direction of its movements.

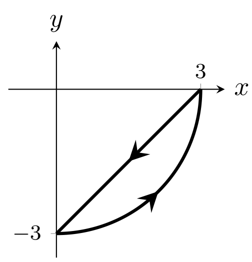
   (b) [9 pts] Parameterize the whale's path. Include appropriate parameter bounds.

   (c) [15 pts] Find the total number (not necessarily an integer) of plankton collected by the whale during its journey on the straight line segment of its path.

   (d) [15 pts] If the ocean current is producing a force field described by $\mathbf{F} = \left(x^2 + y^2\right)\mathbf{i} - 2xy\,\mathbf{j} + 5z\,\mathbf{k}$, find the work done by the force during the whale's swim on the circular portion of the path.

SOLUTION:

(a) Sketch of whale's path in the plane $z = -1$.



(b) Circular portion of path:

$$\mathbf{r}(t) = \langle 3\cos t, 3\sin t, -1\rangle\,, \quad \frac{3\pi}{2} \le t \le 2\pi$$

Straight portion of path:

$$\mathbf{r}(t) = (1-t)\langle 3, 0, -1\rangle + t\langle 0, -3, -1\rangle = \langle 3 - 3t, -3t, -1\rangle\,, \quad 0 \le t \le 1$$

(c) We evaluate the scalar line integral of the density along the line segment.

$$\mathbf{r}'(t) = \langle -3, -3, 0\rangle \implies \|\mathbf{r}'(t)\| = 3\sqrt{2}$$
$$\delta(\mathbf{r}(t)) = (3 - 3t)(-3t)^2(-1 + 2) = 27(t^2 - t^3)$$

$$\text{Total plankton} = \int_C \delta(x, y, z)\,\mathrm{d}s = \int_0^1 \left[27\left(t^2 - t^3\right)(3\sqrt{2})\right]\mathrm{d}t = 81\sqrt{2}\left(\frac{t^3}{3} - \frac{t^4}{4}\right)\Bigg|_0^1 = \frac{27\sqrt{2}}{4}$$

(d) We evaluate the vector line integral of the force along the circular path.

$$\begin{aligned}\mathbf{F}(\mathbf{r}(t)) &= \left\langle (3\cos t)^2 + (3\sin t)^2, -2(3\cos t)(3\sin t), 5(-1)\right\rangle \\ &= \langle 9, -18\cos t\sin t, -5\rangle \\ \mathbf{r}'(t) &= \langle -3\sin t, 3\cos t, 0\rangle \\ \mathbf{F}(\mathbf{r}(t)\cdot\mathbf{r}'(t) &= -27\sin t - 54\cos^2 t\sin t\end{aligned}$$

$$\begin{aligned}\text{Work} = \int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} &= \int_{3\pi/2}^{2\pi}\left(-27\sin t - 54\cos^2 t\sin t\right)\mathrm{d}t \qquad (u = \cos t) \\ &= 27\cos t\Bigg|_{3\pi/2}^{2\pi} + 54\int_0^1 u^2\,\mathrm{d}u \\ &= 27 + \frac{54}{3}u^3\Bigg|_0^1 = 27 + 18 = 45\end{aligned}$$

2. [30 pts] The following problems are not related.

(a) [15 pts] A caterpillar is crawling through the vector force field $\mathbf{F}(x,y) = \left(e^x + y^2\right)\mathbf{i} + \left(e^y + x^2\right)\mathbf{j}$. Its path begins at the origin, follows the line $y = 2x$ to the point where the line intersects the curve $y = \sqrt{8x}$ and returns to the origin along this latter curve. Find the amount of work done by the force field on the caterpillar by evaluating an appropriate double integral.

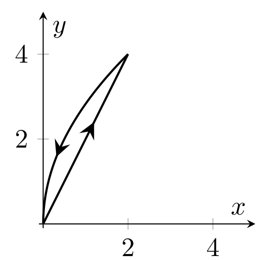
(b) [15 pts] Find the flow of the vector field $\mathbf{F}(x,y) = \left(x^{-2} + y^3\right)\mathbf{i} + \left(3xy^2 - 1\right)\mathbf{j}$ along the path

$$\mathbf{r}(t) = \left(4e^{3t}\cos^2 \pi t\right)\mathbf{i} + \left[e^{-3t}\sin^2\left(\frac{9\pi}{2}t\right)\right]\mathbf{j}, \quad -\frac{1}{3} \le t \le \frac{1}{3}$$

Hint: You don't want to do this directly, but you have a theorem at your side that can make this calculation relatively simple.

SOLUTION:

(a) We will use Green's Theorem. Letting $\mathcal{C}$ be the caterpillar's path, it encloses the region $\mathcal{D}$ as shown in the figure.



The curves intersect when $\sqrt{8x} = 2x \implies 8x = 4x^2 \implies 4x(2-x) = 0 \implies x = 0, 2.$

$$\begin{aligned}
\text{Work} &= \int_{\mathcal{C}} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_{\mathcal{C}} P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}A \\
&= \int_0^2 \int_{2x}^{\sqrt{8x}} \left[\frac{\partial}{\partial x}\left(e^y + x^2\right) - \frac{\partial}{\partial y}\left(e^x + y^2\right)\right]\mathrm{d}y\,\mathrm{d}x = \int_0^2 \int_{2x}^{\sqrt{8x}} (2x - 2y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^2 \left(2xy - y^2\right)\Big|_{2x}^{\sqrt{8x}}\mathrm{d}x = \int_0^2 \left\{2x\sqrt{8x} - (\sqrt{8x})^2 - \left[2x(2x) - (2x)^2\right]\right\}\mathrm{d}x = \int_0^2 \left(4\sqrt{2}x^{3/2} - 8x\right)\mathrm{d}x \\
&= \left(\frac{8\sqrt{2}}{5}x^{5/2} - 4x^2\right)\Bigg|_0^2 = \frac{8\sqrt{2}}{5}2^{5/2} - 16 = \frac{64}{5} - \frac{80}{5} = -\frac{16}{5}
\end{aligned}$$

(b) The path looks ferocious and evaluating $\mathbf{F}$ on the path looks even worse. Maybe we don't need the path at all. Check to see if the vector field is conservative:

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(x^{-2} + y^3\right) = 3y^2 \quad \text{and} \quad \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(3xy^2 - 1\right) = 3y^2 \implies \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

implying that the vector field is conservative. Thus we can choose to find the flow along any path between the endpoints of the curve or, even simpler, use the Fundamental Theorem for Line Integrals. (Note that even using a straight line segment between the two endpoints of the path is not worth the effort)

$$\begin{aligned}
\frac{\partial f}{\partial x} &= x^{-2} + y^3 \implies f(x,y) = \int \left(x^{-2} + y^3\right)\mathrm{d}x = -x^{-1} + xy^3 + g(y) \\
\frac{\partial f}{\partial y} &= 3xy^2 + g'(y) = 3xy^2 - 1 \implies g'(y) = -1 \implies g(y) = -y + C
\end{aligned}$$

Thus the potential function for the vector field is $f(x,y) = xy^3 - y - \dfrac{1}{x} + C$. Now

$$\begin{aligned}
\mathbf{r}\left(-\tfrac{1}{3}\right) &= \left[4e^{3(-1/3)}\cos^2(-\pi/3)\right]\mathbf{i} + \left[e^{-3(-1/3)}\sin^2\left(-9\pi/6\right)\right]\mathbf{j} = e^{-1}\,\mathbf{i} + e\,\mathbf{j} \\
\mathbf{r}\left(\tfrac{1}{3}\right) &= \left[4e^{3(1/3)}\cos^2(\pi/3)\right]\mathbf{i} + \left[e^{-3(1/3)}\sin^2\left(9\pi/6\right)\right]\mathbf{j} = e\,\mathbf{i} + e^{-1}\,\mathbf{j}
\end{aligned}$$

$$\text{Flow} = \int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{\mathcal{C}} \nabla f \cdot \mathrm{d}\mathbf{r} = f\left[\mathbf{r}\left(\tfrac{1}{3}\right)\right] - f\left[\mathbf{r}\left(-\tfrac{1}{3}\right)\right] = f\left(e, e^{-1}\right) - f\left(e^{-1}, e\right)$$

$$= ee^{-3} - e^{-1} - e^{-1} - \left(e^{-1}e^{3} - e - e\right) = e^{-2} - 2e^{-1} - e^{2} + 2e = 2\left(e - e^{-1}\right) + e^{-2} - e^{2}$$

$$= 4\sinh 1 - 2\sinh 2 \quad \text{(just for fun)}$$

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3. [30 pts] Consider the vector field $\mathbf{F} = \langle -y, x, xyz\rangle$ and the surface, $\mathcal{S}$, $z = x^2 + y^2$, $0 \le z \le 1$.

   (a) [15 pts] Calculate the circulation around the boundary of $\mathcal{S}$ with clockwise orientation when viewed from above.

   (b) [15 pts] If possible, verify your calculation in part (a) using any theorem(s) from Calculus III, and clearly state your reasoning! Otherwise, clearly write "Cannot be verified."

SOLUTION:

(a) Parameterize the path as $\mathbf{r}(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k}$, $\;0 \le t \le 2\pi$, giving $\mathbf{r}'(t) = \cos t\,\mathbf{i} - \sin t\,\mathbf{j} + 0\,\mathbf{k}$ and

$$\mathbf{F}(\mathbf{r}(t)) = -\cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \sin t\cos t\,\mathbf{k} \implies \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = -1$$

so that the circulation is given as

$$\text{Circulation} = \int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_0^{2\pi} -1\,\mathrm{d}t = -2\pi$$

(b) We can verify the calculation using Stokes' Theorem with the surface $\mathcal{S}$ being the paraboloid $z = x^2 + y^2$.

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & xyz \end{vmatrix} = xz\,\mathbf{i} - yz\,\mathbf{j} + 2\,\mathbf{k}$$

We project the surface onto the xy–plane, yielding $\mathbf{p} = \mathbf{k}$, the integration region, $\mathcal{R}$, the unit disk, and

$$g(x,y,z) = x^2 + y^2 - z \implies \nabla g = \langle 2x, 2y, -1\rangle \implies |\nabla g \cdot \mathbf{p}| = |-1| = 1$$

Based on the orientation of the path, we use $+\nabla g$ for the normal to the surface giving

$$(\nabla \times \mathbf{F}) \cdot \frac{+\nabla g}{|\nabla g \cdot \mathbf{p}|} = \langle xz, -yz, 2\rangle \cdot \langle 2x, 2y, -1\rangle = 2x^2 z - 2y^2 z - 2 = 2z(x^2 - y^2) - 2$$

and, using the surface to eliminate z, we obtain

$$\text{Circulation} = \iint\limits_{x^2+y^2\le 1} \left[2(x^2+y^2)(x^2-y^2) - 2\right] \mathrm{d}A = 2\int_0^{2\pi}\int_0^1 \left[r^2\left(r^2\cos^2\theta - r^2\sin^2\theta\right) - 1\right] r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= 2\int_0^{2\pi}\int_0^1 \left(r^5\cos 2\theta - r\right)\mathrm{d}r\,\mathrm{d}\theta = 2\left(\int_0^{2\pi}\cos 2\theta\,\mathrm{d}\theta\right)\left(\int_0^1 r^5\,\mathrm{d}r\right) - 2\int_0^{2\pi}\int_0^1 r\,\mathrm{d}r = -2\pi$$

Alternatively, we can use a disk atop the paraboloid as the surface over which to integrate since it shares the boundary with the paraboloid. Projecting onto the xy–plane, we have $\mathbf{p} = \mathbf{k}$, with $\mathcal{R}$, the integration region, the unit disk, and

$$g(x,y,z) = z \implies \nabla g = \langle 0, 0, 1\rangle \implies |\nabla g \cdot \mathbf{p}| = 1$$

For proper orientation we use $-\nabla g$ for the normal to the surface giving

$$(\nabla \times \mathbf{F}) \cdot \frac{-\nabla g}{|\nabla g \cdot \mathbf{p}|} = \langle xz, -yz, 2\rangle \cdot \langle 0, 0, -1\rangle = -2$$

and

$$\text{Circulation} = \iint\limits_{x^2+y^2\le 1} -2\,\mathrm{d}A = -2\iint\limits_{x^2+y^2\le 1} \mathrm{d}A = -2\pi$$

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4. [45 pts] Let $\mathcal{S}$ be the portion of the plane $x + y + z = 1$ in the first octant.

(a) [15 pts] If $\mathcal{S}$ is a thin plate made of a material whose density is $\delta(x,y,z) = x + 2y + 2z$ g/cm^2, what is the mass of $\mathcal{S}$?

(b) [15 pts] Find the downward flux of the vector field $\mathbf{F} = x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}$ through $\mathcal{S}$. [Hint: Feel free to reuse some of the calculations from part (a)]

(c) [15 pts] Let $\mathcal{W}$ be the solid region whose boundary consists of $\mathcal{S}$ and the 3 coordinate planes. Using an appropriate Calculus III theorem, compute the outward flux of $\mathbf{F} = x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\mathbf{k}$ through the boundary of $\mathcal{W}$.

SOLUTION:

(a) We will project the surface onto the xy–plane, giving $\mathbf{p} = \mathbf{k}$ and the integration region $\mathcal{R}$ defined by the inequalities $0 \le x \le 1, 0 \le y \le 1 - x$. Furthermore,

$$g(x,y,z) = x + y + z \implies \nabla g = \langle 1,1,1\rangle \implies \|\nabla g\| = \sqrt{3} \text{ and } |\nabla g \cdot \mathbf{p}| = 1$$

Using the surface to eliminate z gives $\delta = x + 2y + 2z = x + 2y + 2(1 - x - y) = 2 - x$ and we have

$$\text{Mass} = \iint_{\mathcal{S}} \delta(x,y,z)\,\mathrm{d}S = \int_0^1 \int_0^{1-x} (2-x)\sqrt{3}\,\mathrm{d}y\,\mathrm{d}x = \sqrt{3}\int_0^1 (2-x)(1-x)\,\mathrm{d}x$$

$$= \sqrt{3}\int_0^1 \left(2 - 3x + x^2\right)\mathrm{d}x = \sqrt{3}\left(2x - \frac{3}{2}x^2 + \frac{1}{3}x^3\right)\Bigg|_0^1 = \frac{5\sqrt{3}}{6}$$

(b) Most of the work has already been done in part (a). We will use $-\nabla g$ for the downward pointing normal giving

$$\mathbf{F} \cdot \frac{-\nabla g}{|\nabla g \cdot \mathbf{p}|} = \langle x, 2y, 2z\rangle \cdot \frac{\langle -1,-1,-1\rangle}{1} = -x - 2y - 2z = -x - 2y - 2(1 - x - y) = x - 2$$

$$\iint_{\mathcal{S}} \mathbf{F}\cdot \mathrm{d}\mathbf{S} = \iint_{\mathcal{R}} \mathbf{F} \cdot \frac{-\nabla g}{|\nabla g \cdot \mathbf{p}|}\mathrm{d}A = \int_0^1 \int_0^{1-x} (x-2)\,\mathrm{d}y\,\mathrm{d}x = -\frac{5}{6}$$

(c) We use Gauss' Divergence Theorem with $\nabla \cdot \mathbf{F} = 5$. Let $\mathcal{S}_{xy}$, $\mathcal{S}_{yz}$, and $\mathcal{S}_{xz}$ represent the boundaries of the solid $\mathcal{W}$ in the 3 coordinate planes. Then the outward flux through the boundary of $\mathcal{W}$ is

$$\iint_{\text{boundary } \mathcal{W}} \mathbf{F}\cdot \mathrm{d}\mathbf{S} = \iint_{\mathcal{S}_{xy}} \mathbf{F}\cdot \mathrm{d}\mathbf{S} + \iint_{\mathcal{S}_{yz}} \mathbf{F}\cdot \mathrm{d}\mathbf{S} + \iint_{\mathcal{S}_{xy}} \mathbf{F}\cdot \mathrm{d}\mathbf{S} + \iint_{\mathcal{S}} \mathbf{F}\cdot \mathrm{d}\mathbf{S}$$

$$= \iiint_{\mathcal{W}} \nabla \cdot \mathbf{F}\,\mathrm{d}V$$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} 5\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

$$= 5\int_0^1 \int_0^{1-x} (1 - x - y)\,\mathrm{d}y\,\mathrm{d}x = 5\int_0^1 \left[(1-x)y - \frac{1}{2}y^2\right]\Bigg|_0^{1-x} \mathrm{d}x$$

$$= \frac{5}{2}\int_0^1 (1-x)^2\,\mathrm{d}x = -\frac{5}{6}(1-x)^3\Bigg|_0^1 = \frac{5}{6}$$

Note that this is the opposite of the downward flux through the plane as there is no flux of the vector field through the coordinate planes.

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