Write on the front of your bluebook a grading key, your name, student ID, your lecture number and instructor. This exam is worth 150 points and has 4 questions.

- Submit this exam sheet with your bluebook. However, nothing on this exam sheet will be graded. Make sure all of your work is in your bluebook.
- Show all work and simplify your answers! Answers with no justification will receive no points unless otherwise noted. Please begin each problem on a new page.
- You are taking this exam in a proctored and honor code enforced environment. Thus, no notes/papers, calculators, cell phones, or other electronic devices are permitted.

NOTE: Any integrals that need to be evaluated will require integration techniques no more complicated than \( u \)-substitution.

1. [45 pts] A whale is swimming around the ocean collecting plankton in its baleen (like a filter). It swims in the plane \( z = -1 \). Beginning at the point \((x, y, z) = (0, -3, -1)\), it follows a quarter of the circle of radius 3 centered at \((0, 0, -1)\) with \( y \leq 0 \) and \( x \geq 0 \) to the point \((x, y, z) = (3, 0, -1)\). From there it then swims along a straight line to the point where it started. The density of plankton, the food the whale eats, is given by \( \delta(x, y, z) = xy^2(z + 2) \) plankton per meter.

   (a) [6 pts] Sketch the whale’s path in the plane \( z = -1 \), making certain to show the correct direction of its movements.
   (b) [9 pts] Parameterize the whale’s path. Include appropriate parameter bounds.
   (c) [15 pts] Find the total number (not necessarily an integer) of plankton collected by the whale during its journey on the straight line segment of its path.
   (d) [15 pts] If the ocean current is producing a force field described by \( \mathbf{F} = (x^2 + y^2) \mathbf{i} - 2xy \mathbf{j} + 5z \mathbf{k} \), find the work done by the force during the whale’s swim on the circular portion of the path.

2. [30 pts] The following problems are not related.

   (a) [15 pts] A caterpillar is crawling through the vector force field \( \mathbf{F}(x, y) = (e^x + y^2) \mathbf{i} + (e^y + x^2) \mathbf{j} \). Its path begins at the origin, follows the line \( y = 2x \) to the point where the line intersects the curve \( y = \sqrt{8x} \) and returns to the origin along this latter curve. Find the amount of work done by the force field on the caterpillar by evaluating an appropriate double integral.

   (b) [15 pts] Find the flow of the vector field \( \mathbf{F}(x, y) = (x - 2 + y^3) \mathbf{i} + (3xy^2 - 1) \mathbf{j} \) along the path
   \[ \mathbf{r}(t) = (4e^{3t} \cos^2 \pi t) \mathbf{i} + \left[ e^{-3t} \sin^2 \left( \frac{9\pi t}{2} \right) \right] \mathbf{j}, \quad \frac{1}{3} \leq t \leq \frac{1}{3} \]

   Hint: You don’t want to do this directly, but you have a theorem at your side that can make this calculation relatively simple.

3. [30 pts] Consider the vector field \( \mathbf{F} = (-y, x, xyz) \) and the surface, \( S, z = x^2 + y^2, 0 \leq z \leq 1 \).

   (a) [15 pts] Calculate the circulation around the boundary of \( S \) with clockwise orientation when viewed from above.
   (b) [15 pts] If possible, verify your calculation in part (a) using any theorem(s) from Calculus III, and clearly state your reasoning!

   Otherwise, clearly write “Cannot be verified.”

4. [45 pts] Let \( S \) be the portion of the plane \( x + y + z = 1 \) in the first octant.

   (a) [15 pts] If \( S \) is a thin plate made of a material whose density is \( \delta(x, y, z) = x + 2y + 2z \text{ g/cm}^2 \), what is the mass of \( S \)?
   (b) [15 pts] Find the downward flux of the vector field \( \mathbf{F} = x \mathbf{i} + 2y \mathbf{j} + 2z \mathbf{k} \) through \( S \). [Hint: Feel free to reuse some of the calculations from part (a)]
   (c) [15 pts] Let \( W \) be the solid region whose boundary consists of \( S \) and the 3 coordinate planes. Using an appropriate Calculus III theorem, compute the outward flux of \( \mathbf{F} = x \mathbf{i} + 2y \mathbf{j} + 2z \mathbf{k} \) through the boundary of \( W \).

FORMULAS ON BACK
PROJECTION; DISTANCE FROM POINT S TO LINE PARALLEL TO v CONTAINING POINT P; DISTANCE FROM POINT S TO PLANE WITH NORMAL n CONTAINING POINT P

\[
\text{proj}_a b = \left( \frac{a \cdot b}{a \cdot a} \right) a
\]

\[
d = \frac{|F_S \times v|}{|v|} \quad d = \frac{|F_S \cdot n|}{|n|}
\]

ARC LENGTH, FRENET FORMULAS, AND TANGENTIAL AND NORMAL ACCELERATION COMPONENTS

\[
ds = \|v\| \, dt \quad T = \frac{dr}{\|v\|} \quad N = \frac{dT}{ds} \quad B = T \times N
\]

\[
dT
=
\kappa N \quad dB = -\tau N \quad \kappa = \frac{\|v \times a\|}{\|v\|^3} = \frac{|f''(x)|}{\left(1+|f'(x)|^2\right)^{3/2}} \quad \tau = -\frac{\|v\|^2 - a_T}{\|a\|^2}
\]

DIRECTIONAL DERIVATIVE, DISCRIMINANT, AND LAGRANGE MULTIPLIERS

\[
D_A f = \frac{df}{ds} = (\nabla f) \cdot u \quad D = f_{x x} f_{y y} - \left( f_{x y}\right)^2 \quad \nabla f = \lambda \nabla g, \quad \lambda > 0
\]

SECOND DERIVATIVES TEST: Suppose \( f(x, y) \) and its first and second partial derivatives are continuous in a disk centered at \((a, b)\) and \( f_a(a, b) = f_y(a, b) = 0 \).

Let \( D = f_{x x} f_{y y} - \left( f_{x y}\right)^2 \).

- If \( D > 0 \) and \( f_{x x} < 0 \) at \((a, b)\), then \( f \) has a local maximum at \((a, b)\).
- If \( D > 0 \) and \( f_{x x} > 0 \) at \((a, b)\), then \( f \) has a local minimum at \((a, b)\).
- If \( D < 0 \) at \((a, b)\), then \( f \) has a saddle point at \((a, b)\).
- If \( D = 0 \) at \((a, b)\), then the test is inconclusive.

TAYLOR’S FORMULA [at the point \((x_0, y_0)\)]

\[
f(x,y) = f(x_0, y_0) + f_x(x_0, y_0)\left(x-x_0\right) + f_y(x_0, y_0)\left(y-y_0\right) + \frac{1}{2!}\left[f_{x x}(x_0, y_0)\left(x-x_0\right)^2 + 2f_{x y}(x_0, y_0)\left(x-x_0\right)\left(y-y_0\right) + f_{y y}(x_0, y_0)\left(y-y_0\right)^2\right] + \frac{1}{3!}\left[f_{x x x}(x_0, y_0)\left(x-x_0\right)^3 + 3f_{x y y}(x_0, y_0)\left(x-x_0\right)^2\left(y-y_0\right) + 3f_{x y y}(x_0, y_0)\left(x-x_0\right)\left(y-y_0\right)^2 + f_{y y y}(x_0, y_0)\left(y-y_0\right)^3\right] + \cdots
\]

ERROR IN LINEAR APPROXIMATION

\[|E(x,y)| \leq \frac{1}{2!}M \left( |x-x_0| + |y-y_0| \right)^2, \text{ where max } \{(f_{x x}), |f_{x y}|, |f_{y y}| \} \leq M\]

ERROR IN QUADRATIC APPROXIMATION

\[|E(x,y)| \leq \frac{1}{3!}M \left( |x-x_0| + |y-y_0| \right)^3, \text{ where max } \{(f_{x x x}), |f_{x xy}|, |f_{y yy}|, |f_{y y y}| \} \leq M\]

CHANGE OF VARIABLES/SUBSTITUTIONS IN MULTIPLE INTEGRALS

\[
\iint_R f(x,y) \, dA = \iint_S f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \, du \, dv \quad \text{where } J(u,v) = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \left| \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right|
\]

POLAR COORDINATES

\[x = r \cos \theta \quad y = r \sin \theta \quad r^2 = x^2 + y^2 \quad dA = dx \, dy = r \, dr \, d\theta
\]

Coordinate Conversions

\[
\begin{array}{ccc}
\text{Cylindrical to Rectangular} & \text{Spherical to Rectangular} & \text{Spherical to Cylindrical} \\
x = r \cos \theta & x = \rho \sin \phi \cos \theta & r = \rho \sin \phi \\
y = r \sin \theta & y = \rho \sin \phi \sin \theta & y = \rho \cos \phi \\
z = z & z = z & \theta = \theta
\end{array}
\]

\[
dV = dx \, dy \, dz = r \, dr \, d\theta \, dz = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta
\]

MASS, MOMENTS, AND CENTER OF MASS

\[
\begin{array}{ccc}
\text{Mass } M = \iint_R \delta \, dA & \text{Moments } M_x = \iint_R x \, \delta \, dA & \text{Moment } M_y = \iint_R y \, \delta \, dA & \text{Center of mass } \overrightarrow{\tau} = \frac{M_x}{M} \quad \overrightarrow{\eta} = \frac{M_y}{M}
\end{array}
\]

FLOW AND FLUX

Flow = \( \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{dr} = \int_C P \, dx + Q \, dy \)

Flux = \( \int_C \mathbf{F} \cdot n \, ds = \int_C P \, dy - Q \, dx \)

\[
\mathbf{n} = \mathbf{T} \times \mathbf{k} \quad \mathbf{F}(x,y) = P(x,y) \mathbf{i} + Q(x,y) \mathbf{j}
\]

GREEN’S THEOREM

\[
\text{C is the boundary curve enclosing the region } D, \text{ traversed counterclockwise.}
\]

CIRCULATION/TANGENTIAL FORM

\[
\int_C \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, ds = \iint_D P \, dx + Q \, dy
\]

FLUX/NORMAL FORM

\[
\iint_D \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \, dA = \oint_C P \, dy - Q \, dx
\]

SURFACE AREA OF LEVEL SURFACE \( g(x,y,z) = c \)

\[
S = \iint_S ds = \iint_R \frac{||\nabla g||}{|\nabla g \cdot \mathbf{p}|} \, dA
\]

STOKES’ THEOREM

\[
\int_C (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r} \quad C \text{ is the boundary curve of the surface } S
\]

GAUSS’ DIVERGENCE THEOREM

\[
\iiint_E \nabla \cdot \mathbf{F} \, dV = \iint_S \mathbf{F} \cdot d\mathbf{S} \quad S \text{ is the boundary surface of the solid } E
\]

FUN TRIGONOMETRY FACTS

\[
\sin^2 x = \frac{1 - \cos 2x}{2} \quad \cos^2 x = \frac{1 + \cos 2x}{2}
\]