Write on the front of your bluebook a grading key, your name, student ID, your lecture number and instructor. This exam is worth 150 points and has 4 questions.

- Submit this exam sheet with your bluebook. However, nothing on this exam sheet will be graded. Make sure all of your work is in your bluebook.
- **Show all work and simplify your answers!** Answers with no justification will receive no points unless otherwise noted. **Please begin each problem on a new page.**
- You are taking this exam in a proctored and honor code enforced environment. Thus, no notes/papers, calculators, cell phones, or other electronic devices are permitted.

NOTE: Any integrals that need to be evaluated will require integration techniques no more complicated than *u*-substitution.

1. [45 pts] A whale is swimming around the ocean collecting plankton in its baleen (like a filter). It swims in the plane $z = -1$. Beginning at the point $(x, y, z) = (0, -3, -1)$, it follows a quarter of the circle of radius 3 centered at $(0, 0, -1)$ with $y \leq 0$ and $x \geq 0$ to the point $(x, y, z) = (3, 0, -1)$. From there it then swims along a straight line to the point where it started. The density of plankton, the food the whale eats, is given by $\delta(x, y, z) = xy^2(z + 2)$ plankton per meter.
   (a) [6 pts] Sketch the whale's path in the plane $z = -1$, making certain to show the correct direction of its movements.
   (b) [9 pts] Parameterize the whale's path. Include appropriate parameter bounds.
   (c) [15 pts] Find the total number (not necessarily an integer) of plankton collected by the whale during its journey on the straight line segment of its path.
   (d) [15 pts] If the ocean current is producing a force field described by $\mathbf{F} = \left(x^2 + y^2\right)\mathbf{i} - 2xy\,\mathbf{j} + 5z\,\mathbf{k}$, find the work done by the force during the whale's swim on the circular portion of the path.
2. [30 pts] The following problems are not related.
   (a) [15 pts] A caterpillar is crawling through the vector force field $\mathbf{F}(x, y) = \left(e^x + y^2\right)\mathbf{i} + \left(e^y + x^2\right)\mathbf{j}$. Its path begins at the origin, follows the line $y = 2x$ to the point where the line intersects the curve $y = \sqrt{8x}$ and returns to the origin along this latter curve. Find the amount of work done by the force field on the caterpillar by evaluating an appropriate double integral.
   (b) [15 pts] Find the flow of the vector field $\mathbf{F}(x, y) = \left(x^{-2} + y^3\right)\mathbf{i} + \left(3xy^2 - 1\right)\mathbf{j}$ along the path
   $$\mathbf{r}(t) = \left(4e^{3t}\cos^2 \pi t\right)\mathbf{i} + \left[e^{-3t}\sin^2\left(\frac{9\pi}{2}t\right)\right]\mathbf{j}, \quad -\frac{1}{3} \leq t \leq \frac{1}{3}$$
   Hint: You don't want to do this directly, but you have a theorem at your side that can make this calculation relatively simple.
3. [30 pts] Consider the vector field $\mathbf{F} = \langle -y, x, xyz\rangle$ and the surface, $\mathcal{S}$, $z = x^2 + y^2$, $0 \leq z \leq 1$.
   (a) [15 pts] Calculate the circulation around the boundary of $\mathcal{S}$ with clockwise orientation when viewed from above.
   (b) [15 pts] If possible, verify your calculation in part (a) using any theorem(s) from Calculus III, and clearly state your reasoning! Otherwise, clearly write "Cannot be verified."
4. [45 pts] Let $\mathcal{S}$ be the portion of the plane $x + y + z = 1$ in the first octant.
   (a) [15 pts] If $\mathcal{S}$ is a thin plate made of a material whose density is $\delta(x, y, z) = x + 2y + 2z$ g/cm^2, what is the mass of $\mathcal{S}$?
   (b) [15 pts] Find the downward flux of the vector field $\mathbf{F} = x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}$ through $\mathcal{S}$. [Hint: Feel free to reuse some of the calculations from part (a)]
   (c) [15 pts] Let $\mathcal{W}$ be the solid region whose boundary consists of $\mathcal{S}$ and the 3 coordinate planes. Using an appropriate Calculus III theorem, compute the outward flux of $\mathbf{F} = x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\mathbf{k}$ through the boundary of $\mathcal{W}$.

FORMULAS ON BACK

PROJECTION; DISTANCE FROM POINT S TO LINE PARALLEL TO $\mathbf{v}$ CONTAINING POINT P; DISTANCE FROM POINT S TO PLANE WITH NORMAL $\mathbf{n}$ CONTAINING POINT P

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a} \qquad d = \frac{\left\|\overrightarrow{PS}\times\mathbf{v}\right\|}{\|\mathbf{v}\|} \qquad d = \left|\overrightarrow{PS}\cdot\frac{\mathbf{n}}{\|\mathbf{n}\|}\right|$$

ARC LENGTH, FRENET FORMULAS, AND TANGENTIAL AND NORMAL ACCELERATION COMPONENTS

$$\mathrm{d}s = \|\mathbf{v}\|\,\mathrm{d}t \qquad \mathbf{T} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = \frac{\mathbf{v}}{\|\mathbf{v}\|} \qquad \mathbf{N} = \frac{\mathrm{d}\mathbf{T}/\mathrm{d}s}{\|\mathrm{d}\mathbf{T}/\mathrm{d}s\|} = \frac{\mathrm{d}\mathbf{T}/\mathrm{d}t}{\|\mathrm{d}\mathbf{T}/\mathrm{d}t\|} \qquad \mathbf{B} = \mathbf{T}\times\mathbf{N}$$

$$\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} = \kappa\mathbf{N} \qquad \frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s} = -\tau\mathbf{N} \qquad \kappa = \left\|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\right\| = \frac{\|\mathbf{v}\times\mathbf{a}\|}{\|\mathbf{v}\|^3} = \frac{|f''(x)|}{\left\{1+[f'(x)]^2\right\}^{3/2}} = \frac{|\dot{x}\ddot{y}-\dot{y}\ddot{x}|}{(\dot{x}^2+\dot{y}^2)^{3/2}} \qquad \tau = -\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s}\cdot\mathbf{N}$$

$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N} \qquad a_T = \frac{\mathrm{d}\|\mathbf{v}\|}{\mathrm{d}t} \qquad a_N = \kappa\|\mathbf{v}\|^2 = \sqrt{\|\mathbf{a}\|^2 - a_T^2}$$

DIRECTIONAL DERIVATIVE, DISCRIMINANT, AND LAGRANGE MULTIPLIERS $\quad D_{\mathbf{u}}f = \dfrac{\mathrm{d}f}{\mathrm{d}s} = (\nabla f)\cdot\mathbf{u} \qquad D = f_{xx}f_{yy} - (f_{xy})^2 \qquad \nabla f = \lambda\nabla g, \quad g = 0$

SECOND DERIVATIVES TEST: Suppose $f(x,y)$ and its first and second partial derivatives are continuous in a disk centered at (a,b) and $f_x(a,b) = f_y(a,b) = 0$. Let $D = f_{xx}f_{yy} - f_{xy}^2$.

- If $D > 0$ and $f_{xx} < 0$ at (a,b), then f has a local maximum at (a,b).
- If $D > 0$ and $f_{xx} > 0$ at (a,b), then f has a local minimum at (a,b).
- If $D < 0$ at (a,b), then f has a saddle point at (a,b).
- If $D = 0$ at (a,b), then the test is inconclusive.

TAYLOR'S FORMULA [at the point (x_0, y_0)]

$$f(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + \frac{1}{2!}\left[f_{xx}(x_0,y_0)(x-x_0)^2 + 2f_{xy}(x_0,y_0)(x-x_0)(y-y_0) + f_{yy}(x_0,y_0)(y-y_0)^2\right]$$
$$+ \frac{1}{3!}\left[f_{xxx}(x_0,y_0)(x-x_0)^3 + 3f_{xxy}(x_0,y_0)(x-x_0)^2(y-y_0) + 3f_{xyy}(x_0,y_0)(x-x_0)(y-y_0)^2 + f_{yyy}(x_0,y_0)(y-y_0)^3\right] + \cdots$$

ERROR IN LINEAR APPROXIMATION $\quad |E(x,y)| \le \dfrac{1}{2!}M\left(|x-x_0| + |y-y_0|\right)^2$, where $\max\left\{|f_{xx}|, |f_{xy}|, |f_{yy}|\right\} \le M$

ERROR IN QUADRATIC APPROXIMATION $\quad |E(x,y)| \le \dfrac{1}{3!}M\left(|x-x_0| + |y-y_0|\right)^3$, where $\max\left\{|f_{xxx}|, |f_{xxy}|, |f_{xyy}|, |f_{yyy}|\right\} \le M$

CHANGE OF VARIABLES/SUBSTITUTIONS IN MULTIPLE INTEGRALS

$$\iint_{\mathcal{R}} f(x,y)\,\mathrm{d}A = \iint_{\mathcal{S}} f(x(u,v), y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\mathrm{d}u\,\mathrm{d}v \quad \text{where} \quad J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

POLAR COORDINATES $\quad x = r\cos\theta \qquad y = r\sin\theta \qquad r^2 = x^2 + y^2 \qquad \mathrm{d}A = \mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$

Coordinate Conversions

Cylindrical to Rectangular	Spherical to Rectangular	Spherical to Cylindrical
$x = r\cos\theta$	$x = \rho\sin\phi\cos\theta$	$r = \rho\sin\phi$
$y = r\sin\theta$	$y = \rho\sin\phi\sin\theta$	$z = \rho\cos\phi$
$z = z$	$z = \rho\cos\phi$	$\theta = \theta$

$$\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z = \rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$$

MASS, MOMENTS, AND CENTER OF MASS $\quad$ Mass $M = \iint_R \delta\,\mathrm{d}A \quad$ Moments $M_x = \iint_R y\,\delta\,\mathrm{d}A \quad M_y = \iint_R x\,\delta\,\mathrm{d}A \quad$ Center of mass $\overline{x} = \dfrac{M_y}{M} \quad \overline{y} = \dfrac{M_x}{M}$

FLOW AND FLUX $\quad$ Flow $= \int_C \mathbf{F}\cdot\mathbf{T}\,\mathrm{d}s = \int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_C P\,\mathrm{d}x + Q\,\mathrm{d}y \quad$ Flux $= \int_C \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}s = \int_C P\,\mathrm{d}y - Q\,\mathrm{d}x \quad \mathbf{n} = \mathbf{T}\times\mathbf{k} \quad \mathbf{F}(x,y) = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}$

GREEN'S THEOREM $\quad \mathcal{C}$ is the boundary curve enclosing the region $\mathcal{D}$, traversed counterclockwise.

CIRCULATION/TANGENTIAL FORM $\quad \displaystyle\iint_{\mathcal{D}}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}A = \oint_{\mathcal{C}} P\,\mathrm{d}x + Q\,\mathrm{d}y \qquad$ FLUX/NORMAL FORM $\quad \displaystyle\iint_{\mathcal{D}}\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)\mathrm{d}A = \oint_{\mathcal{C}} P\,\mathrm{d}y - Q\,\mathrm{d}x$

SURFACE AREA OF LEVEL SURFACE $g(x,y,z) = c \quad S = \displaystyle\iint_{\mathcal{S}}\mathrm{d}S = \iint_{\mathcal{R}}\frac{\|\nabla g\|}{|\nabla g\cdot\mathbf{p}|}\,\mathrm{d}A$

STOKES' THEOREM $\quad \displaystyle\iint_{\mathcal{S}}(\boldsymbol{\nabla}\times\mathbf{F})\cdot\mathrm{d}\mathbf{S} = \oint_{\mathcal{C}}\mathbf{F}\cdot\mathrm{d}\mathbf{r} \qquad \mathcal{C}$ is the boundary curve of the surface $\mathcal{S}$

GAUSS' DIVERGENCE THEOREM $\quad \displaystyle\iiint_{\mathcal{E}}\boldsymbol{\nabla}\cdot\mathbf{F}\,\mathrm{d}V = \iint_{\mathcal{S}}\mathbf{F}\cdot\mathrm{d}\mathbf{S} \qquad \mathcal{S}$ is the boundary surface of the solid $\mathcal{E}$

FUN TRIGONOMETRY FACTS $\quad \sin^2 x = \dfrac{1-\cos 2x}{2} \qquad \cos^2 x = \dfrac{1+\cos 2x}{2}$