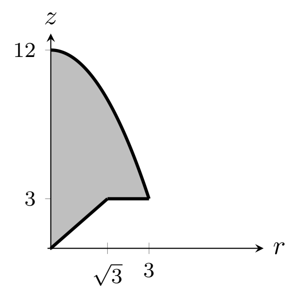
NOTE: Any integrals that need to be evaluated will require integration techniques no more complicated than u-substitution.

1. [32 pts] You are in charge of making three-dimensional stoppers for glass bottles. The density of the material from which the stoppers are made is $\delta(x, y, z) = x + y + 7$. A cross section of half of the stopper is depicted in the following rz–plane (constant θ plane). The curved portion of the stopper is $x^2 + y^2 + z = 12$.



   (a) [15 pts] Set up, **but do not evaluate**, the appropriate integral(s) to find the mass of the stopper using cylindrical coordinates and the order $\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$.

   (b) [17 pts] Now suppose that only the curved portion of the stopper is replaced with a portion of the sphere of radius 3, centered at $(x, y, z) = (0, 0, 3)$, while the rest of the stopper's shape remains the same.

      i. [2 pts] Make a sketch of this new stopper in the rz–plane (constant θ plane). Be sure to label important points such as intercepts.

      ii. [15 pts] Using spherical coordinates and the order $\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$, set up, **but do not evaluate**, the appropriate integral(s) to find the mass of this new stopper.

SOLUTION:

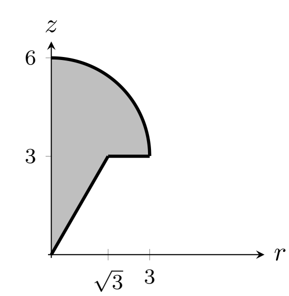
(a)

$$\text{Mass} = \int_0^{2\pi}\int_0^{\sqrt{3}}\int_{\sqrt{3}r}^{12-r^2} (r\cos\theta + r\sin\theta + 7)\, r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta + \int_0^{2\pi}\int_{\sqrt{3}}^{3}\int_{3}^{12-r^2} (r\cos\theta + r\sin\theta + 7)\, r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$$

Alternate answer

$$\text{Mass} = \int_0^{2\pi}\int_0^{\sqrt{3}}\int_{\sqrt{3}r}^{3} (r\cos\theta + r\sin\theta + 7)\, r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta + \int_0^{2\pi}\int_0^{3}\int_{3}^{12-r^2} (r\cos\theta + r\sin\theta + 7)\, r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$$

(b) i. Sketch of new region.



ii. The sphere that forms the new top of the stopper has spherical coordinates equation

$$x^2 + y^2 + (z-3)^2 = 9 \implies x^2 + y^2 + z^2 - 6z + 9 = 9 \implies \rho^2 - 6\rho\cos\phi = 0 \implies \rho = 6\cos\phi$$

Thus

$$\text{Mass} = \int_0^{2\pi}\int_0^{\pi/6}\int_0^{6\cos\phi} \left[\rho\sin\phi\,(\cos\theta + \sin\theta) + 7\right]\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$$
$$+ \int_0^{2\pi}\int_{\pi/6}^{\pi/4}\int_{3\sec\phi}^{6\cos\phi} \left[\rho\sin\phi\,(\cos\theta + \sin\theta) + 7\right]\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$$

Alternate answer

$$\text{Mass} = \int_0^{2\pi}\int_0^{\pi/6}\int_0^{3\sec\phi} \left[\rho\sin\phi\,(\cos\theta+\sin\theta)+7\right]\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$$
$$+\int_0^{2\pi}\int_0^{\pi/4}\int_{3\sec\phi}^{6\cos\phi} \left[\rho\sin\phi\,(\cos\theta+\sin\theta)+7\right]\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$$

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2. [20 pts] The charge density on a metal plate is given by $q(x,y)=\sqrt{y/x}+\sqrt{xy}$ coulombs per square meter. The plate occupies the first quadrant region bounded by $y=1/x$, $y=9/x$, $y=x$, and $y=4x$. Using an appropriate change of variables from xy–space to uv–space, find the total charge on the plate.

SOLUTION:

The boundaries of the plate can be written in a more revealing way as $xy=1$, $xy=9$, $y/x=1$ and $y/x=4$. This shows that the boundary of the plate consists of two sets of "parallel" curves, suggesting the change of variables $u=xy$ and $v=y/x$. Then multiplying these together yields $uv=y^2 \implies y=\sqrt{uv}=u^{1/2}v^{1/2}$ with division giving $x^2=u/v \implies x=\sqrt{u/v}=u^{1/2}v^{-1/2}$. Then

$$J(u,v)=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{1}{2}u^{-1/2}v^{-1/2} & -\frac{1}{2}u^{1/2}v^{-3/2}\\ \frac{1}{2}u^{-1/2}v^{1/2} & \frac{1}{2}u^{1/2}v^{-1/2}\end{vmatrix}=\frac{1}{4v}+\frac{1}{4v}=\frac{1}{2v}$$

The original region of integration is given by $1\le xy\le 9$ and $1\le y/x\le 4$ giving the new region of integration as $1\le u\le 9$ and $1\le v\le 4$ and $q(x(u,v),y(u,v))=\sqrt{v}+\sqrt{u}$. Thus, noting that $\frac{1}{2v}>0$ on the new region of integration,

$$\begin{aligned}\text{Total charge} &= \int_1^9\int_1^4\left(\sqrt{v}+\sqrt{u}\right)\left|\frac{1}{2v}\right|\mathrm{d}v\,\mathrm{d}u=\frac{1}{2}\int_1^9\int_1^4\left(v^{-1/2}+u^{1/2}v^{-1}\right)\mathrm{d}v\,\mathrm{d}u\\ &=\frac{1}{2}\int_1^9\left(2v^{1/2}+u^{1/2}\ln|v|\right)\Big|_1^4\,\mathrm{d}u=\frac{1}{2}\int_1^9\left(2+u^{1/2}\ln 4\right)\mathrm{d}u\\ &=\frac{1}{2}\left(2u+\frac{2}{3}u^{3/2}\ln 4\right)\Big|_1^9=8+\frac{52}{3}\ln 2\text{ coulombs}\end{aligned}$$

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3. [28 pts] The following problems are not related.

   (a) [10 pts] In a recent insect population study, an entomologist has told you that the number of mosquitoes per unit volume in a certain region is given by $\rho(x,y,z)=100\sqrt{y}, y\ge 0$. You are trying to trap some of the mosquitoes and have built an enclosure, $\mathcal{W}$, located in the first octant of a Cartesian coordinate system. The roof of the enclosure is $z=\sin y^2$ for $0\le y\le\sqrt{\pi}$ with walls given by $y=x^2$, $x=0$ and $y=\sqrt{\pi}$. How many mosquitoes can you expect to find in the enclosure?

   (b) [18 pts] Consider the following triple integral: $\displaystyle\int_{-\sqrt{3}}^{\sqrt{3}}\int_{-\sqrt{3-y^2}}^{\sqrt{3-y^2}}\int_{2-\sqrt{4-x^2-y^2}}^{\sqrt{(x^2+y^2)/3}}\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$

      i. [3 pts] Make a well-labeled sketch in the zr–plane (constant θ) depicting the region of integration.

      ii. [12 pts] Convert the integral to spherical coordinates and evaluate the result.

      iii. [3 pts] Provide a brief description of what the triple integral represents geometrically.

SOLUTION:

(a) The total number of mosquitoes will be given by the integral of the density over the region $\mathcal{W}$:

$$\text{Total Mosquitoes} = \iiint_{\mathcal{W}} \rho(x,y,z)\,\mathrm{d}V = \int_0^{\sqrt[4]{\pi}} \int_{x^2}^{\sqrt{\pi}} \int_0^{\sin y^2} 100\sqrt{y}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = 100 \int_0^{\sqrt[4]{\pi}} \int_{x^2}^{\sqrt{\pi}} \sqrt{y}\,z\Big|_0^{\sin y^2} \mathrm{d}y\,\mathrm{d}x$$

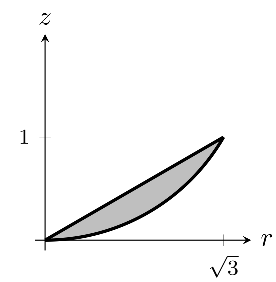
$$= 100 \int_0^{\sqrt[4]{\pi}} \int_{x^2}^{\sqrt{\pi}} \sqrt{y}\sin y^2\,\mathrm{d}y\,\mathrm{d}x \quad \text{(switch order of integration)}$$

$$= 100 \int_0^{\sqrt{\pi}} \int_0^{\sqrt{y}} \sqrt{y}\sin y^2\,\mathrm{d}x\,\mathrm{d}y = 100 \int_0^{\sqrt{\pi}} \sqrt{y}\sin y^2\,x\Big|_0^{\sqrt{y}} \mathrm{d}y$$

$$= 100 \int_0^{\sqrt{\pi}} y\sin y^2\,\mathrm{d}y \quad (u = y^2)$$

$$= 50 \int_0^{\pi} \sin u\,\mathrm{d}u = 50(-\cos u)\Big|_0^{\pi} = 100$$

(b) i. Sketch of the region.



ii. The x– and y–bounds are the circle of radius 3 centered at the origin. Conversion of the lower z–bound to spherical coordinates gives

$$\begin{aligned} & z = 2 - \sqrt{4 - x^2 - y^2} \\ \Longrightarrow\ & \rho\cos\phi - 2 = -\sqrt{4 - \rho^2\sin^2\phi} \\ \Longrightarrow\ & \rho^2\cos^2\phi - 4\rho\cos\phi + 4 = 4 - \rho^2\sin^2\phi \\ \Longrightarrow\ & \rho^2 = 4\rho\cos\phi \\ \Longrightarrow\ & \rho = 4\cos\phi \end{aligned}$$

while conversion of the upper z–bound to spherical coordinates yields

$$z = \sqrt{\frac{x^2+y^2}{3}} \implies \rho\cos\phi = \frac{\rho\sin\phi}{\sqrt{3}} \implies \tan\phi = \sqrt{3} \implies \phi = \frac{\pi}{3}$$

$$\int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-y^2}}^{\sqrt{3-y^2}} \int_{2-\sqrt{4-x^2-y^2}}^{\sqrt{(x^2+y^2)/3}} \mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y = \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \int_0^{4\cos\phi} \rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$$

$$= \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \frac{1}{3}\rho^3\sin\phi\Big|_0^{4\cos\phi} \mathrm{d}\phi\,\mathrm{d}\theta$$

$$= \frac{64}{3} \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \cos^3\phi\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta \quad (u = \cos\phi)$$

$$= \frac{64}{3} \int_0^{2\pi} \int_0^{1/2} u^3\,\mathrm{d}u\,\mathrm{d}\theta = \frac{16}{3} \int_0^{2\pi} u^4\Big|_0^{1/2} \mathrm{d}\theta$$

$$= \frac{1}{3} \int_0^{2\pi} \mathrm{d}\theta = \frac{2\pi}{3}$$

iii. The integral determines the volume of the region above the sphere of radius 2 centered at $(0,0,2)$ and below the cone $z = \sqrt{\frac{x^2+y^2}{3}}$.

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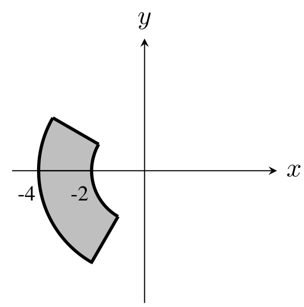
4. [20 pts] The bottom of a lake is described by the function $s(x, y) = \dfrac{-1}{1 + (x/2)^2 + (y/2)^2}$. The lake's center is at the origin of a Cartesian coordinate system whose positive y–axis points north with the lake's surface at $z = 0$. In order to conduct some underwater experiments, your lab supervisor needs to know the average depth of that part of the lake lying to the west of the lake's center, bounded by the lines $y = \sqrt{3}x$ and $y = -x/\sqrt{3}$ and the circular arcs 2 and 4 units from the center of the lake.

(a) [4 pts] Make a sketch in the plane $z = 0$ of that part of the lake whose average depth you are required to find.

(b) [16 pts] What number should you report to the lab supervisor?

SOLUTION:

(a) The region of the lake at $z = 0$ is the domain $\mathcal{D}$ over which will need to perform the integration. Here is a sketch of $\mathcal{D}$.



(b) The integrand and the region of integration are screaming for polar coordinates. To handle this, the region $\mathcal{D}$ of integration is

$$2 \le r \le 4 \quad \text{and} \quad \frac{5\pi}{6} \le \theta \le \frac{4\pi}{3}$$

and the integrand becomes

$$s(x, y) = \frac{-1}{1 + (x/2)^2 + (y/2)^2} = \frac{-1}{1 + (x^2 + y^2)/4} = \frac{-1}{1 + r^2/4} = s(r, \theta)$$

We need the area of the region,

$$\text{Area}(\mathcal{D}) = \int_{5\pi/6}^{4\pi/3} \int_2^4 r \, \mathrm{d}r \, \mathrm{d}\theta = \left(\int_{5\pi/6}^{4\pi/3} \mathrm{d}\theta \right) \left(\int_2^4 r \, \mathrm{d}r \right) = \left(\frac{4\pi}{3} - \frac{5\pi}{6} \right) \left(\frac{r^2}{2} \bigg|_2^4 \right) = 3\pi$$

Furthermore,

$$\iint_{\mathcal{D}} s(x, y) \, \mathrm{d}A = \int_{5\pi/6}^{4\pi/3} \int_2^4 \frac{-1}{1 + r^2/4} r \, \mathrm{d}r \, \mathrm{d}\theta \qquad (u = 1 + r^2/4)$$

$$= \left(\int_{5\pi/6}^{4\pi/3} \mathrm{d}\theta \right) \left(-2 \int_2^5 \frac{\mathrm{d}u}{u} \right) = -\pi \ln \frac{5}{2}$$

Thus,

$$s_{\text{average}} = \frac{1}{\text{Area}(\mathcal{D})} \iint_{\mathcal{D}} s(x, y) \, \mathrm{d}A = \frac{-\pi \ln(5/2)}{3\pi} = -\frac{1}{3} \ln \frac{5}{2}$$

is the number to report to the lab supervisor.

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