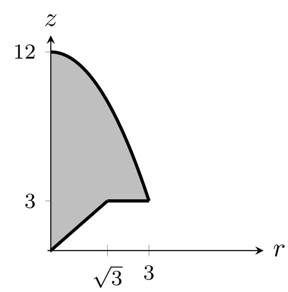
Write on the front of your bluebook a grading key, your name, student ID, your lecture number and instructor. This exam is worth 100 points and has 4 questions.

- Submit this exam sheet with your bluebook. However, nothing on this exam sheet will be graded. Make sure all of your work is in your bluebook.
- **Show all work and simplify your answers!** Answers with no justification will receive no points unless otherwise noted. **Please begin each problem on a new page.**
- You are taking this exam in a proctored and honor code enforced environment. Thus, no notes/papers, calculators, cell phones, or other electronic devices are permitted.

NOTE: Any integrals that need to be evaluated will require integration techniques no more complicated than u-substitution.

1. [32 pts] You are in charge of making three-dimensional stoppers for glass bottles. The density of the material from which the stoppers are made is $\delta(x, y, z) = x + y + 7$. A cross section of half of the stopper is depicted in the following rz–plane (constant θ plane). The curved portion of the stopper is $x^2 + y^2 + z = 12$.



   (a) [15 pts] Set up, **but do not evaluate**, the appropriate integral(s) to find the mass of the stopper using cylindrical coordinates and the order $\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$.

   (b) [17 pts] Now suppose that only the curved portion of the stopper is replaced with a portion of the sphere of radius 3, centered at $(x, y, z) = (0, 0, 3)$, while the rest of the stopper's shape remains the same.

      i. [2 pts] Make a sketch of this new stopper in the rz–plane (constant θ plane). Be sure to label important points such as intercepts.

      ii. [15 pts] Using spherical coordinates and the order $\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$, set up, **but do not evaluate**, the appropriate integral(s) to find the mass of this new stopper.

2. [20 pts] The charge density on a metal plate is given by $q(x, y) = \sqrt{y/x} + \sqrt{xy}$ coulombs per square meter. The plate occupies the first quadrant region bounded by $y = 1/x$, $y = 9/x$, $y = x$, and $y = 4x$. Using an appropriate change of variables from xy–space to uv–space, find the total charge on the plate.

3. [28 pts] The following problems are not related.

   (a) [10 pts] In a recent insect population study, an entomologist has told you that the number of mosquitoes per unit volume in a certain region is given by $\rho(x, y, z) = 100\sqrt{y}, y \geq 0$. You are trying to trap some of the mosquitoes and have built an enclosure, $\mathcal{W}$, located in the first octant of a Cartesian coordinate system. The roof of the enclosure is $z = \sin y^2$ for $0 \leq y \leq \sqrt{\pi}$ with walls given by $y = x^2$, $x = 0$ and $y = \sqrt{\pi}$. How many mosquitoes can you expect to find in the enclosure?

   (b) [18 pts] Consider the following triple integral: $\displaystyle\int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-y^2}}^{\sqrt{3-y^2}} \int_{2-\sqrt{4-x^2-y^2}}^{\sqrt{(x^2+y^2)/3}} \mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$

      i. [3 pts] Make a well-labeled sketch in the zr–plane (constant θ) depicting the region of integration.

      ii. [12 pts] Convert the integral to spherical coordinates and evaluate the result.

      iii. [3 pts] Provide a brief description of what the triple integral represents geometrically.

CONTINUED ON BACK

4. [20 pts] The bottom of a lake is described by the function $s(x,y) = \dfrac{-1}{1+(x/2)^2+(y/2)^2}$. The lake's center is at the origin of a Cartesian coordinate system whose positive y–axis points north with the lake's surface at $z=0$. In order to conduct some underwater experiments, your lab supervisor needs to know the average depth of that part of the lake lying to the west of the lake's center, bounded by the lines $y=\sqrt{3}x$ and $y=-x/\sqrt{3}$ and the circular arcs 2 and 4 units from the center of the lake.

(a) [4 pts] Make a sketch in the plane $z=0$ of that part of the lake whose average depth you are required to find.

(b) [16 pts] What number should you report to the lab supervisor?

PROJECTION; DISTANCE FROM POINT S TO LINE PARALLEL TO $\mathbf{v}$ CONTAINING POINT P; DISTANCE FROM POINT S TO PLANE WITH NORMAL $\mathbf{n}$ CONTAINING POINT P

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a} \qquad d = \frac{\left\|\overrightarrow{PS}\times\mathbf{v}\right\|}{\|\mathbf{v}\|} \qquad d = \left|\overrightarrow{PS}\cdot\frac{\mathbf{n}}{\|\mathbf{n}\|}\right|$$

ARC LENGTH, FRENET FORMULAS, AND TANGENTIAL AND NORMAL ACCELERATION COMPONENTS

$$\mathrm{d}s = \|\mathbf{v}\|\,\mathrm{d}t \qquad \mathbf{T} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = \frac{\mathbf{v}}{\|\mathbf{v}\|} \qquad \mathbf{N} = \frac{\mathrm{d}\mathbf{T}/\mathrm{d}s}{\|\mathrm{d}\mathbf{T}/\mathrm{d}s\|} = \frac{\mathrm{d}\mathbf{T}/\mathrm{d}t}{\|\mathrm{d}\mathbf{T}/\mathrm{d}t\|} \qquad \mathbf{B} = \mathbf{T}\times\mathbf{N}$$

$$\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} = \kappa\mathbf{N} \qquad \frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s} = -\tau\mathbf{N} \qquad \kappa = \left\|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\right\| = \frac{\|\mathbf{v}\times\mathbf{a}\|}{\|\mathbf{v}\|^3} = \frac{|f''(x)|}{\left\{1+[f'(x)]^2\right\}^{3/2}} = \frac{|\dot{x}\ddot{y}-\dot{y}\ddot{x}|}{(\dot{x}^2+\dot{y}^2)^{3/2}} \qquad \tau = -\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s}\cdot\mathbf{N}$$

$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N} \qquad a_T = \frac{\mathrm{d}\|\mathbf{v}\|}{\mathrm{d}t} \qquad a_N = \kappa\|\mathbf{v}\|^2 = \sqrt{\|\mathbf{a}\|^2 - a_T^2}$$

DIRECTIONAL DERIVATIVE, DISCRIMINANT, AND LAGRANGE MULTIPLIERS $\quad D_{\mathbf{u}}f = \dfrac{\mathrm{d}f}{\mathrm{d}s} = (\nabla f)\cdot\mathbf{u} \qquad D = f_{xx}f_{yy} - (f_{xy})^2 \qquad \nabla f = \lambda\nabla g, \quad g = 0$

CHANGE OF VARIABLES/SUBSTITUTIONS IN MULTIPLE INTEGRALS

$$\iint_{\mathcal{R}} f(x,y)\,\mathrm{d}A = \iint_{\mathcal{S}} f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\mathrm{d}u\,\mathrm{d}v \quad \text{where} \quad J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

POLAR COORDINATES $\quad x = r\cos\theta \qquad y = r\sin\theta \qquad r^2 = x^2+y^2 \qquad \mathrm{d}A = \mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$

Coordinate Conversions

Cylindrical to Rectangular	Spherical to Rectangular	Spherical to Cylindrical
$x = r\cos\theta$	$x = \rho\sin\phi\cos\theta$	$r = \rho\sin\phi$
$y = r\sin\theta$	$y = \rho\sin\phi\sin\theta$	$z = \rho\cos\phi$
$z = z$	$z = \rho\cos\phi$	$\theta = \theta$

$$\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z = \rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$$

MASS, MOMENTS, AND CENTER OF MASS $\quad$ Mass $M = \iint_R \delta\,\mathrm{d}A \quad$ Moments $M_x = \iint_R y\,\delta\,\mathrm{d}A \quad M_y = \iint_R x\,\delta\,\mathrm{d}A \quad$ Center of mass $\overline{x} = \dfrac{M_y}{M} \quad \overline{y} = \dfrac{M_x}{M}$