1. [24 pts] The following problems are not related.

(a) Plane P_1 is given by $x - y + z = 1$ and plane P_2 is given by $x + y + z = 1$. Find the projection of the normal vector of P_1 onto the normal vector of P_2.

(b) You are playing around with your new laser pointer inside your house. Part of the ceiling can be described by $x - y + 4z = 24$. If the beam from the laser pointer travels along the line whose symmetric equations are

$$x - 1 = \frac{y+1}{3} = \frac{z-2}{4},$$

will the beam hit the ceiling? If so, find the coordinates of the point where the beam shines on the ceiling. If not, explain why not.

(c) Find the equation of the plane containing the points $P_1(-2, 1, 4)$ and $P_2(1, 0, 3)$ that is perpendicular to the plane $4x - y + 3z = 2$. Write your answer in the form $ax + by + cz = d$.

SOLUTION:

(a) The normal to P_1 is $\mathbf{n}_1 = \langle 1, -1, 1\rangle$ and the normal to P_2 is $\mathbf{n}_2 = \langle 1, 1, 1\rangle$.

$$\text{proj}_{\mathbf{n_2}}\mathbf{n}_1 = \left(\frac{\mathbf{n}_1 \cdot \mathbf{n_2}}{\mathbf{n}_2 \cdot \mathbf{n_2}}\right)\mathbf{n}_2 = \frac{\langle 1,-1,1\rangle \cdot \langle 1,1,1\rangle}{\langle 1,1,1\rangle \cdot \langle 1,1,1\rangle}\langle 1,1,1\rangle = \left\langle \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\rangle$$

(b) To see if the beam intersects the ceiling, we need to see if there exists a t such that the coordinates of the line, $(x(t), y(t), z(t))$ satisfy the equation of the plane. With

$$x(t) = 1 + t, \quad y(t) = -1 + 3t, \quad z(t) = 2 + 4t$$

we have

$$\begin{aligned} 1 + t - (-1 + 3t) + 4(2 + 4t) &= 24 \\ 10 + 14t &= 24 \\ t &= 1 \end{aligned}$$

So the beam does hit the ceiling at the point $(x(1), y(1), z(1)) = (2, 2, 6)$.

(c) A vector in the plane of interest is $\overrightarrow{P_1P_2} = \langle 1 - (-2), 0 - 1, 3 - 4\rangle = \langle 3, -1, -1\rangle$. The plane we seek needs to be perpendicular to the given plane, implying that the given plane's normal, $\langle 4, -1, 3\rangle$ is parallel to the plane we seek. The cross product of these two vectors will serve as the normal to the sought after plane.

$$\langle 3,-1,-1\rangle \times \langle 4,-1,3\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & -1 \\ 4 & -1 & 3 \end{vmatrix} = -4\,\mathbf{i} - 13\,\mathbf{j} + \mathbf{k}$$

The equation of the plane is then (we can use either point in the plane here)

$$-4(x-1) - 13(y-0) + 1(z-3) = 0 \implies -4x - 13y + z = -1 \text{ or } 4x + 13y - z = 1$$

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2. [24 pts] The following problems are related.

(a) Find an equation of the surface consisting of all points that are equidistant from the point $(1, -2, 0)$ and the plane $y = -4$, simplifying your answer.

(b) Identify the surface.

(c) Find the equation of the curve of intersection of the surface and the yz–plane. Simplify your answer.

SOLUTION:

(a) Let (x, y, z) be an arbitrary point on the surface. Then the distance from the given point to this arbitrary point is

$$d_1 = \sqrt{(x-1)^2 + (y+2)^2 + z^2}.$$

A point in the plane is $P(0,-4,0)$, with a vector from that point to the arbitrary point $S(x,y,z)$ on the surface given by $\overrightarrow{PS} = \langle x, y+4, z\rangle$. The normal to the plane is **j** with $\|\mathbf{j}\| = 1$ so the distance from the arbitrary point to the plane is

$$d_2 = \left| \langle x, y+4, z\rangle \cdot \frac{\langle 0,1,0\rangle}{1} \right| = |y+4|$$

Equating these and simplifying yields

$$\begin{aligned} d_1 &= d_2 \\ \sqrt{(x-1)^2+(y+2)^2+z^2} &= |y+4| \\ (x-1)^2+(y+2)^2+z^2 &= (y+4)^2 \\ (x-1)^2+z^2+y^2+4y+4 &= y^2+8y+16 \\ y &= \frac{1}{4}\left[(x-1)^2+z^2\right]-3 \end{aligned}$$

(b) Circular paraboloid with vertex at $(1,-3,0)$ and axis parallel to the y–axis, opening toward the positive y–axis.

(c) On the yz–plane $x=0$ so the curve of intersection is

$$y = \frac{1}{4}\left[(-1)^2+z^2\right]-3 = \frac{1}{4}z^2 - \frac{11}{4}$$

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3. [40 pts] A satellite orbits a planet (whose center is located at the origin) along the trajectory (path)

$$\mathbf{r}(t) = \cos(\pi t)\ \mathbf{i} + \sqrt{3}\sin(\pi t)\ \mathbf{j} + \sqrt{2}\cos(\pi t)\ \mathbf{k}$$

where t is in hours and x, y, z are in kilometers.

(a) Calculate the speed of the satellite.

(b) Calculate $\mathbf{T}(t)$ and $\mathbf{N}(t)$ for the satellite's trajectory.

(c) Calculate the curvature of the satellite's path.

(d) What are the tangential and normal components of the acceleration of the satellite?

(e) At time $t=3$ hours, the satellite detects an asteroid straight ahead at a distance of 4 km. Find the coordinates of the asteroid.

SOLUTION:

(a)

$$\begin{aligned} \mathbf{r}'(t) &= -\pi\sin(\pi t)\,\mathbf{i} + \pi\sqrt{3}\cos(\pi t)\,\mathbf{j} - \pi\sqrt{2}\sin(\pi t)\,\mathbf{k} \\ \implies \|\mathbf{r}'(t)\| &= \sqrt{\pi^2\sin^2(\pi t)+3\pi^2\cos^2(\pi t)+2\pi^2\sin^2(\pi t)} = \sqrt{3}\pi \end{aligned}$$

(b)

$$\begin{aligned} \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} &= \frac{1}{\sqrt{3}\pi}\left(-\pi\sin(\pi t)\,\mathbf{i} + \pi\sqrt{3}\cos(\pi t)\,\mathbf{j} - \pi\sqrt{2}\sin(\pi t)\,\mathbf{k}\right) \\ &= \frac{1}{\sqrt{3}}\left(-\sin(\pi t)\,\mathbf{i} + \sqrt{3}\cos(\pi t)\,\mathbf{j} - \sqrt{2}\sin(\pi t)\,\mathbf{k}\right) \end{aligned}$$

Now

$$\begin{aligned} \mathbf{T}'(t) &= \frac{1}{\sqrt{3}}\left(-\pi\cos(\pi t)\,\mathbf{i} - \sqrt{3}\pi\sin(\pi t)\,\mathbf{j} - \sqrt{2}\pi\cos(\pi t)\,\mathbf{k}\right) \\ \implies \|\mathbf{T}'(t)\| &= \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2\left(\pi^2\cos^2(\pi t)+3\pi^2\sin^2(\pi t)+2\pi^2\cos^2(\pi t)\right)} = \pi \\ \implies \mathbf{N}(t) &= \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{1}{\sqrt{3}}\left(-\cos(\pi t)\,\mathbf{i} - \sqrt{3}\sin(\pi t)\,\mathbf{j} - \sqrt{2}\cos(\pi t)\,\mathbf{k}\right) \end{aligned}$$

(c)

$$\mathbf{r}''(t) = -\pi^2\cos(\pi t)\,\mathbf{i} - \pi^2\sqrt{3}\sin(\pi t)\,\mathbf{j} - \pi^2\sqrt{2}\cos(\pi t)\,\mathbf{k}$$

$$\begin{aligned}\mathbf{r}'(t)\times\mathbf{r}''(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\pi\sin(\pi t) & \pi\sqrt{3}\cos(\pi t) & -\pi\sqrt{2}\sin(\pi t) \\ -\pi^2\cos(\pi t) & -\pi^2\sqrt{3}\sin(\pi t) & -\pi^2\sqrt{2}\cos(\pi t) \end{vmatrix} \\ &= \left[-\sqrt{6}\pi^3\cos^2(\pi t) - \sqrt{6}\pi^3\sin^2(\pi t)\right]\mathbf{i} + \left[\sqrt{2}\pi^3\sin(\pi t)\cos(\pi t) - \sqrt{2}\pi^3\sin(\pi t)\cos(\pi t)\right]\mathbf{j} \\ &\qquad \left[\sqrt{3}\pi^3\sin^2(\pi t) + \sqrt{3}\pi^3\cos^2(\pi t)\right]\mathbf{k} \\ &= -\sqrt{6}\pi^3\,\mathbf{i} + \sqrt{3}\pi^3\mathbf{k} \end{aligned}$$

$$\implies \|\mathbf{r}'(t)\times\mathbf{r}''(t)\| = \sqrt{6\pi^6 + 3\pi^6} = 3\pi^3$$

Thus,

$$\kappa = \frac{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{3\pi^3}{\left(\sqrt{3}\pi\right)^3} = \frac{1}{\sqrt{3}}$$

Note that the curvature can also be calculated as

$$\kappa = \left\|\frac{\mathrm{d}\mathbf{T}}{\mathbf{d}s}\right\| = \left\|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s}\right\| = \left\|\frac{\mathrm{d}\mathbf{T}/\mathrm{d}t}{\mathrm{d}s/\mathrm{d}t}\right\| = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\pi}{\sqrt{3}\pi} = \frac{1}{\sqrt{3}}$$

(d)

$$a_T = \frac{\mathrm{d}\|\mathbf{v}\|}{\mathrm{d}t} = \frac{\mathrm{d}\sqrt{3}\pi}{\mathrm{d}t} = 0$$

$$a_N = \kappa\|\mathbf{v}\|^2 = \frac{1}{\sqrt{3}}\left(\sqrt{3}\pi\right)^2 = \sqrt{3}\pi^2$$

(e) Since the asteroid is straight ahead of the satellite, it is in the direction of $\mathbf{T}(3) = -\mathbf{j}$. The satellite is at $\mathbf{r}(3) = -\mathbf{i} - \sqrt{2}\,\mathbf{k}$. Thus, the coordinates of the satellite are

$$\mathbf{r}(3) + 4\mathbf{T}(3) = \left(-\mathbf{i} - \sqrt{2}\,\mathbf{k}\right) - 4\,\mathbf{j} = -\mathbf{i} - 4\,\mathbf{j} - \sqrt{2}\,\mathbf{k} \quad \text{or} \quad \left(-1, -4, -\sqrt{2}\right)$$

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4. [12 pts] In your blue book, write **TRUE** if the statement is true or write **FALSE** if the statement is false. No justification required and no partial credit given.

(a) $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$

(b) The curvature of a path is a measure of how the unit tangent vector changes as one moves along the path.

(c) If $\mathbf{u}$ and $\mathbf{v}$ are three dimensional vectors, then $|\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\|\mathbf{v}\|$

(d) If a particle moves along a smooth path in the plane $x = -7$, then its unit binormal vector $\mathbf{B}$ has only a $\mathbf{j}$–component.

(e) If $\mathbf{u}$ and $\mathbf{v}$ are two nonzero vectors such that $\mathbf{u}\times\mathbf{v} = \mathbf{0}$, then there exists a real number c such that $\mathbf{u} = c\mathbf{v}$.

(f) If $\dfrac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = -t\,\mathbf{i} - t\,\mathbf{j} - t\,\mathbf{k}$ and $\mathbf{r}(1) = \mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}$, then $\mathbf{r}(t) = \frac{1}{2}\left[(3-t^2)\mathbf{i} + (5-t^2)\mathbf{j} + (7-t^2)\mathbf{k}\right]$

SOLUTION:

(a) **FALSE** - $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$

(b) **TRUE** - $\kappa = \left\|\dfrac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\right\|$

(c) **TRUE** - $|\mathbf{u}\cdot\mathbf{v}| = \|\mathbf{u}\|\|\mathbf{v}\|\,|\cos\theta| \le |\mathbf{u}\|\|\mathbf{v}\|$ since $|\cos\theta| \le 1$

(d) **FALSE** - The osculating plane is $x = -7$ and the normal to that plane is $\pm\mathbf{i}$ so $\mathbf{B}$ has only an $\mathbf{i}$–component.

(e) **TRUE** - The cross product vanishing implies that the angle between the two vectors is 0, implying that the vectors are parallel, meaning that they are scalar multiples of one another.

(f) **TRUE** - Note that $\mathbf{r}(1) = \frac{1}{2}(2\,\mathbf{i} + 4\,\mathbf{j} + 6\,\mathbf{k}) = \mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}$ and $\mathbf{r}'(t) = -t\,\mathbf{i} - t\,\mathbf{j} - t\,\mathbf{k}$

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