APPM 2350, Summer 2018: Final Exam July 27, 2018

INSTRUCTIONS: Please show all of your work and make your methods and reasoning clear. Answers out of the blue with no supporting work will receive no credit (unless the directions to a specific problem say otherwise, of course). No calculators or electronic devices are allowed. Please start each numbered problem on a new page in your bluebook and do the problems in order. Please sign the front of your blue book indicating you read and understood these directions in addition to the CU honor code.

SIMPLIFY ALL ANSWERS AS FAR AS POSSIBLE!!!

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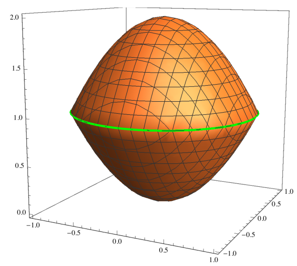
1. (25 points) Find the outward flux of $\mathbf{F}(x,y,z) = \langle 4x, x^2-2y, 3z+x^2\rangle$ across the surface of the solid E bounded by $z = x^2+y^2$ and $z = 2-x^2-y^2$.

Solution: The solid looks something like this, it's trapped between two paraboloids:



where the green curve of intersection can be found by solving

$$x^2+y^2 = 2-x^2-y^2 \iff 2x^2+2y^2 = 2 \iff x^2+y^2 = 1,$$

which is a circle of radius 1, centered on the z-axis, but in the plane $z = 1$ (since $z = x^2+y^2$).

Note that $\operatorname{div}\mathbf{F} = 4-2+3 = 5$, so by the Gauss' Theorem the outward flux across the surface σ of E is

$$\iint_\sigma \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_E \operatorname{div}\mathbf{F}\,dV = 5\iiint_E 1\,dV = 5\text{vol}(E).$$

To actually find this volume we should do the integral in cylindrical coordinates, where the projection of E onto the xy-plane is the unit circular disk R given by $x^2+y^2 \le 1$. So

$$\begin{aligned}
\iint_\sigma \mathbf{F}\cdot\mathbf{n}\,dS &= 5\iint_R \int_{x^2+y^2}^{2-x^2-y^2} 1\,dz\,dA \\
&= 5\int_0^{2\pi}\int_0^1\int_{r^2}^{2-r^2} r\,dz\,dr\,d\theta \\
&= 5\cdot 2\pi\int_0^1 r\left[z\right]_{r^2}^{2-r^2} \\
&= 10\pi\int_0^1 r(2-r^2-r^2)\,dr \\
&= 10\pi\int_0^1 (2r-2r^3)\,dr \\
&= 10\pi\left[r^2-\frac{1}{2}r^4\right]_0^1 \\
&= 10\pi\cdot\frac{1}{2} \\
&= \boxed{5\pi}
\end{aligned}$$

2. (15 points) A few unrelated questions. You don't have to show any work in this problem.

(a) The integral

$$\int_0^{2\pi} \int_0^3 \int_r^{\sqrt{18-r^2}} r\, dz\, dr\, d\theta$$

represents the volume of a solid E in $\mathbb{R}^3$. Write the integrals in spherical coordinates, in the order $d\rho\, d\phi\, d\theta$, that also represent the volume of the same solid E. You don't have to actually find the volume of E.

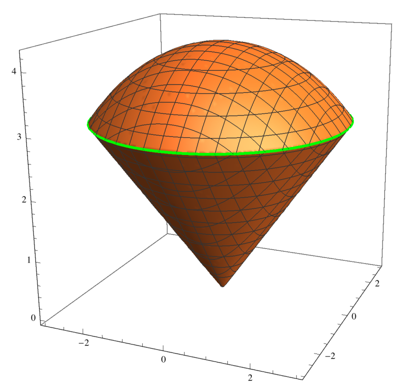
(b) Find the positively oriented simple closed curve C for which the value of the line integral

$$\int_C (y^3 - y)\, dx - 2x^3\, dy$$

is a maximum.

Solution:

(a) The solid E looks something like this:



It's trapped between the cone $z = r$ and the upper hemisphere $z = \sqrt{18 - r^2}$. Set them equal to each other to find the curve of intersection $r = 3$, a circle of radius 3 in the plane $z = 3$. Then

$$\boxed{\text{vol}(E) = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sqrt{18}} \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta}$$

(b) By Green's Theorem

$$\int_C (y^3 - y)\, dx - 2x^3\, dy = \iint_R \left[-6x^2 - \left(3y^2 - 1\right)\right]\, dA = \iint_R \left(1 - 6x^2 - 3y^2\right)\, dA$$

where R is the region in the xy-plane bounded by the curve C. This double integral will be largest when the integrand is nonnegative, that is, when

$$1 - 6x^2 - 3y^2 \geq 0 \qquad \text{or} \qquad 6x^2 + 3y^2 \leq 1,$$

an elliptical disk in the xy-plane. So we should choose C to be the boundary curve of this disk, which is the ellipse $\boxed{6x^2 + 3y^2 = 1}$

3. (25 points) Consider the vector field $\mathbf{F}(x, y, z) = \langle z^2 - x, 2y, z^2 xy \rangle$ and the curve of intersection C, oriented counterclockwise when viewed from above, of the paraboloid $z = x^2 + y^2$ and the plane $z = 4$.

(a) Using an appropriate theorem, provide *two* different integrals that both equal the circulation of **F** around C, a line integral and a surface integral. To save time you don't have to actually evaluate the integrals, but you should simplify/transform the integrals you provide in your answer, meaning you should provide a regular single integral with respect to time and a regular double integral with respect to area of simplified, scalar-valued functions, with clear limits/region of integration.

(b) It turns out that if you do the integrals in part (a) you get zero. So, is **F** conservative?

Solution:

(a) To setup the line integral for circulation $\oint_C \mathbf{F} \cdot d\mathbf{r}$, we first parameterize the curve C with

$$\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), 4 \rangle, \quad 0 \le t \le 2\pi, \qquad \Longrightarrow \qquad \mathbf{r}'(t) = \langle -2\sin(t), 2\cos(t), 0 \rangle$$

Then

$$\begin{aligned}
\oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\
&= \int_0^{2\pi} \langle 16 - 2\cos(t), 4\sin(t), 64\cos(t)\sin(t) \rangle \cdot \langle -2\sin(t), 2\cos(t), 0 \rangle\, dt \\
&= \int_0^{2\pi} -32\sin(t) + 4\sin(t)\cos(t) + 8\sin(t)\cos(t)\, dt \\
&= \boxed{\int_0^{2\pi} -32\sin(t) + 12\sin(t)\cos(t)\, dt}
\end{aligned}$$

As for the surface integral, we want $\iint_\sigma \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$, where σ is the paraboloid $z = x^2 + y^2$, $0 \le z \le 4$, oriented *upwards*. Write $z - x^2 - y^2 = 0$ so that

$$g(x, y, z) = z - x^2 - y^2 \qquad \Longrightarrow \qquad \nabla g = \langle -2x, -2y, 1 \rangle$$

and we use this instead of its opposite due to the orientation. We project onto the xy-plane so that $\mathbf{p} = \mathbf{k} \implies |\nabla g \cdot \mathbf{p}| = 1$. The region R in the xy-plane is the circular disk $x^2 + y^2 \le 4$.

You should find that

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \langle xz^2, 2z - yz^2, 0 \rangle$$

Evaluating $\operatorname{curl} \mathbf{F}$ on the surface σ, we have

$$\begin{aligned}
\iint_\sigma \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} &= \iint_R \langle x(x^2+y^2)^2, 2(x^2+y^2) - y(x^2+y^2)^2, 0 \rangle \cdot \frac{\langle -2x, -2y, 1 \rangle}{1}\, dA \\
&= \boxed{\iint_R -2x^2(x^2+y^2)^2 - 4y(x^2+y^2) + 2y^2(x^2+y^2)^2\, dA}
\end{aligned}$$

and that's good enough. Note that you could convert to polar coordinates to evaluate this integral without much trouble. Note also there is a much easier way to set up a correct flux integral. Do you see it?

(b). $\boxed{\text{No}}$, as in part (a), $\operatorname{curl} \mathbf{F} \neq \mathbf{0}$ so **F** cannot be conservative.

4. (35 points) Consider the vector field $\mathbf{F}(x,y) = \langle 4x+y, 4y+x \rangle$.

(a) Explain briefly why $\mathbf{F}$ is conservative.

(b) Find a potential function $\varphi(x,y)$ for $\mathbf{F}$. Don't get this part wrong – the rest of the problem depends on it! But you *shouldn't* get this wrong since you can check your answer easily.

(c) Use Lagrange multipliers to find the absolute extrema (both max and min) of φ subject to the constraint $x^2+y^2=1$. Be sure to find both the point(s) where the absolute extrema occur *and* what the absolute extreme *values* are. [Hint: you should find 4 points total. I won't tell you which points yield maximum or minimum values, but I will tell you that there is one point in each quadrant in the xy-plane. Call these points P_1, P_2, P_3 and P_4, where the subscript is the quadrant the point is in.]

(d) Suppose a particle starts at P_1 and goes to P_2 along a line segment, then to P_3, then to P_4 and then back to P_1, always along line segments. Find the work done by $\mathbf{F}$ on the particle as it travels along this quadrilateral path. You might want to sketch a graph.

(e) If you were standing at the point P_1 on the graph of $z=\varphi(x,y)$, in what direction should you head to maximize the rate of change of φ with respect to distance? What is this maximum rate of change?

Solution:

(a) Set $P(x,y) = 4x+y$ and $Q(x,y) = 4y+x$. Then $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1-1 = 0$. Also, the domain of $\mathbf{F}$ is all of $\mathbb{R}^2$ so is simply connected. Therefore $\mathbf{F}$ is conservative.

(b) One way or another you should get $\varphi(x,y) = 2x^2+xy+2y^2$.

(c) Solve the system of equations $\nabla\varphi = \lambda\nabla g$ and $g(x,y)=1$ where $g(x,y) = x^2+y^2$. That is, solve the system

$$\langle 4x+y, 4y+x \rangle = \lambda\langle 2x, 2y \rangle, \quad x^2+y^2=1$$

or

$$4x+y = 2\lambda x, \quad 4y+x = 2\lambda y, \quad x^2+y^2=1.$$

Solve for λ in the first two equations:

$$\lambda = 2+\frac{y}{2x} \quad \text{and} \quad \lambda = 2+\frac{x}{2y},$$

where we have divided by $2x$ and $2y$ so must consider what happens when $x=0$ or $y=0$. If x were zero, the first equation above would force $y=0$ too, but this doesn't satisfy the constraint $x^2+y^2=1$. Likewise, if $y=0$, then the second equation forces $x=0$, and these values don't satisfy the constraint either. So we may safely proceed assuming $x \neq 0$ and $y \neq 0$. Setting $\lambda=\lambda$ as above yields

$$2+\frac{y}{2x} = 2+\frac{x}{2y} \quad \text{or} \quad \frac{y}{2x} = \frac{x}{2y} \quad \text{or} \quad 2y^2 = 2x^2 \quad \text{or} \quad x^2=y^2.$$

Substituting this into the constraint yields

$$x^2+x^2=1 \implies x = \pm\sqrt{\frac{1}{2}}.$$

Now, if $x=\sqrt{1/2}$, then $y^2 = 1/2 \implies y = \pm\sqrt{1/2}$. This gives two points of interest $(\sqrt{1/2}, \pm\sqrt{1/2})$. Similarly, if $x = -\sqrt{1/2}$, we get $y = \pm\sqrt{1/2}$ so there are two more points of interest: $(-\sqrt{1/2}, \pm\sqrt{1/2})$. So

$$P_1 = (\sqrt{1/2}, \sqrt{1/2}), \quad P_2 = (-\sqrt{1/2}, \sqrt{1/2}), \quad P_3 = (-\sqrt{1/2}, -\sqrt{1/2}), \quad P_4 = (\sqrt{1/2}, -\sqrt{1/2})$$

Note that

$$\varphi(P_1) = \varphi(P_3) = 5/2 \qquad \text{and} \qquad \varphi(P_2) = \varphi(P_4) = 3/2$$

so φ has an absolute maximum value of $5/2$ at P_1 and P_3 and an absolute minimum value of $3/2$ at P_2 and P_4 subject to the constraint $x^2 + y^2 = 1$.

(d) The work done is zero since this quadrilateral path (it's a square, of course) is closed and **F** is conservative.

(e) You should head in the direction of the gradient at P_1:

$$\nabla\varphi(P_1) = \mathbf{F}(P_1) = \langle 4\sqrt{1/2} + \sqrt{1/2}, 4\sqrt{1/2} + \sqrt{1/2}\rangle = \boxed{\langle 5\sqrt{1/2}, 5\sqrt{1/2}\rangle}$$

The maximum rate of change at P_1 in this direction is

$$\begin{aligned} |\nabla\varphi(P_1)| &= \sqrt{(5\sqrt{1/2})^2 + (5\sqrt{1/2})^2} \\ &= \sqrt{25/2 + 25/2} \\ &= \sqrt{25} \\ &= \boxed{5} \end{aligned}$$