1. (25 points) Find the outward flux of \( \mathbf{F}(x, y, z) = \langle 4x, x^2 - 2y, 3z + x^2 \rangle \) across the surface of the solid \( E \) bounded by \( z = x^2 + y^2 \) and \( z = 2 - x^2 - y^2 \).

2. (15 points) A few unrelated questions. You don’t have to show any work in this problem.

   (a) The integral
   \[
   \int_0^{2\pi} \int_0^3 \int_0^{\sqrt{18-r^2}} r \, dz \, dr \, d\theta
   \]
   represents the volume of a solid \( E \) in \( \mathbb{R}^3 \). Write the integrals in spherical coordinates, in the order \( d\rho \, d\phi \, d\theta \), that also represent the volume of the same solid \( E \). You don’t have to actually find the volume of \( E \).

   (b) Find the positively oriented simple closed curve \( C \) for which the value of the line integral
   \[
   \int_C (y^3 - y) \, dx - 2x^3 \, dy
   \]
   is a maximum.

3. (25 points) Consider the vector field \( \mathbf{F}(x, y, z) = \langle z^2 - x, 2y, z^2 \cdot xy \rangle \) and the curve of intersection \( C \), oriented counterclockwise when viewed from above, of the paraboloid \( z = x^2 + y^2 \) and the plane \( z = 4 \).

   (a) Using an appropriate theorem, provide two different integrals that both equal the circulation of \( \mathbf{F} \) around \( C \), a line integral and a surface integral. To save time you don’t have to actually evaluate the integrals, but you should simplify/transform the integrals you provide in your answer, meaning you should provide a regular single integral with respect to time and a regular double integral with respect to area of simplified, scalar-valued functions, with clear limits/region of integration.

   (b) It turns out that if you do the integrals in part (a) you get zero. So, is \( \mathbf{F} \) conservative?

OVER FOR MORE PROBLEMS!
4. (35 points) Consider the vector field $\mathbf{F}(x, y) = (4x + y, 4y + x)$.

(a) Explain briefly why $\mathbf{F}$ is conservative.

(b) Find a potential function $\varphi(x, y)$ for $\mathbf{F}$. Don’t get this part wrong – the rest of the problem depends on it! But you shouldn’t get this wrong since you can check your answer easily.

(c) Use Lagrange multipliers to find the absolute extrema (both max and min) of $\varphi$ subject to the constraint $x^2 + y^2 = 1$. Be sure to find both the point(s) where the absolute extrema occur and what the absolute extreme values are. [Hint: you should find 4 points total. I won’t tell you which points yield maximum or minimum values, but I will tell you that there is one point in each quadrant in the $xy$-plane. Call these points $P_1, P_2, P_3$ and $P_4$, where the subscript is the quadrant the point is in.]

(d) Suppose a particle starts at $P_1$ and goes to $P_2$ along a line segment, then to $P_3$, then to $P_4$ and then back to $P_1$, always along line segments. Find the work done by $\mathbf{F}$ on the particle as it travels along this quadrilateral path. You might want to sketch a graph.

(e) If you were standing at the point $P_1$ on the graph of $z = \varphi(x, y)$, in what direction should you head to maximize the rate of change of $\varphi$ with respect to distance? What is this maximum rate of change?

\[
\text{proj}_\mathbf{a} \mathbf{b} = \left( \mathbf{a} \cdot \mathbf{b} \right) \mathbf{b}, \quad d = \frac{|\mathbf{F} \times \mathbf{v}|}{|\mathbf{v}|}, \quad d = \left| \frac{\mathbf{F} \cdot \mathbf{n}}{|\mathbf{n}|} \right|, \quad \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}, \quad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}, \quad \mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)
\]

\[
\frac{d\mathbf{T}}{ds} = \kappa \mathbf{N}, \quad \kappa(t) = \left| \frac{\mathbf{T}'(t)}{|\mathbf{r}'(t)|} \right| = \left| \frac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{|\mathbf{r}'(t)|^3} \right|, \quad \kappa(x) = \frac{|f''(x)|}{\sqrt{1 + (f'(x))^2}}, \quad \frac{d\mathbf{B}}{ds} = -\tau \mathbf{N}
\]

\[
\sin(2\theta) = 2 \sin(\theta) \cos(\theta), \quad \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta), \quad \cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta)), \quad \sin^2(\theta) = \frac{1}{2}(1 - \sin(2\theta))
\]

\[
\frac{df}{ds} = \nabla f \cdot \mathbf{u} \quad D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - (f_{xy}(x, y))^2 \quad \nabla f = \lambda \nabla g, \quad g = 0
\]

\[
f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) + \frac{1}{2!} \left[ f_{xx}(a, b)(x - a)^2 + 2f_{xy}(a, b)(x - a)(y - b) + f_{yy}(y - b)^2 \right] + \frac{1}{3!} \left[ f_{xxx}(a, b)(x - a)^3 + 3f_{xxy}(a, b)(x - a)^2(y - b) + 3f_{xyy}(a, b)(x - a)(y - b)^2 + f_{yyy}(a, b)(y - b)^3 \right]
\]

\[n^{th} \text{ order Taylor approximation error centered at the point } (a, b): \]

\[
|E(x, y)| \leq \frac{M}{(n + 1)!} (|x - a| + |y - b|)^{n+1}
\]

where $M$ is an upper bound on the absolute value of all the relevant $n + 1^{st}$ order partial derivatives of $f$.

\[
J(u, v) = \left| \begin{array}{cc}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}
\end{array} \right| \quad x = \rho \sin(\phi) \cos(\theta), y = \rho \sin(\phi) \sin(\theta), z = \rho \cos(\phi)
\]

outward flux across a curve: \[
\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_R \nabla \cdot \mathbf{F} \, dA \quad \text{where } \mathbf{n} = \mathbf{T} \times \mathbf{k}
\]

Stokes: \[
\oint_C \mathbf{F} \cdot \mathbf{T} \, ds = \iint_S \nabla \times \mathbf{F} \, dS \\
\text{Gauss': } \iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_E \nabla \cdot \mathbf{F} \, dV
\]