APPM 2350, Summer 2018: Exam 3 July 13, 2018

INSTRUCTIONS: Please show all of your work and make your methods and reasoning clear. Answers out of the blue with no supporting work will receive no credit (unless the directions to a specific problem say otherwise, of course). No calculators or electronic devices are allowed. Please start each numbered problem on a new page in your bluebook and do the problems in order. Please sign the front of your blue book indicating you read and understood these directions in addition to the CU honor code.

SIMPLIFY ALL ANSWERS AS FAR AS POSSIBLE!!!

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1. (20 points) The integral

$$\int_0^4 \int_{\sqrt{y}}^2 1\,dx\,dy$$

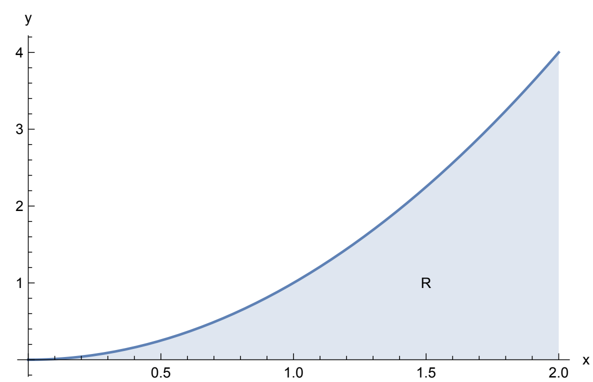
represents the area of a thin, flat plate R in the xy-plane.

(a) Sketch a graph of the region R, labeling anything of interest including the axes.

(b) Find the mass of the plate given the density function $\rho(x,y) = \sin(x^3)$.

Solution:

(a)

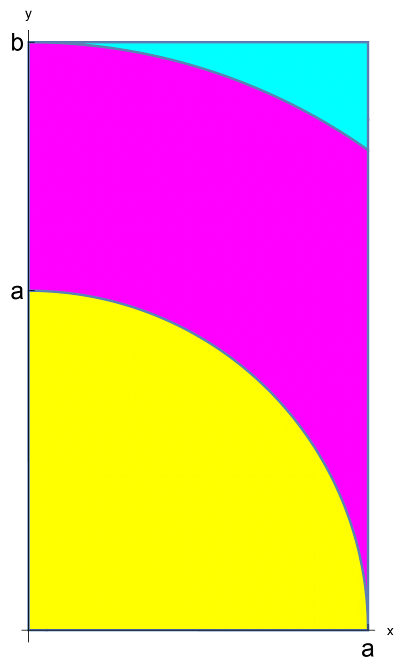


(b) Switch the order of integration:

$$\begin{aligned}
\text{mass} &= \iint_R \rho(x,y)\,dA \\
&= \int_0^2 \int_0^{x^2} \sin(x^3)\,dy\,dx \\
&= \int_0^2 \sin(x^3)\Big[y\Big]_0^{x^2}\,dx \\
&= \frac{1}{3}\int_0^2 \sin(x^3)3x^2\,dx \\
&= -\frac{1}{3}\Big[\cos(x^3)\Big]_0^2 \\
&= -\frac{1}{3}\left(\cos(8) - \cos(0)\right) \\
&= \frac{1}{3}\left(1 - \cos(8)\right)
\end{aligned}$$

2. (24 points) Consider the rectangular region R in the first quadrant in the xy-plane bounded by $x = 0$, $y = 0$, $x = a$ and $y = b$ where a and b are positive constants with $a < b$. Please set up the double integral(s) in polar coordinates in the order $d\theta\, dr$ that equal the area of R. You don't have to actually do the integral(s), just set them up! (The area is ab of course, but that's not the question!) Include a relevant sketch as part of your answer.

Solution: Sketch the region first, it should look like this, where we've color-coded various strips corresponding to relevant ranges of r-values.



Note that the righthand boundary line $x = a$ is $r\cos(\theta) = a$, or writing θ in terms of r is $\theta = \arccos(a/r)$ (note $r > 0$ in this strip). Similarly, the upper boundary line $y = b$ is $r\sin(\theta) = b \implies \theta = \arcsin(b/r)$. Also, the farthest distance away from the origin in the whole region is the upper righthand corner, which is $\sqrt{a^2 + b^2}$ units away from $(0, 0)$.

We need three integrals corresponding to the green, blue and red regions (in that order here) like this:

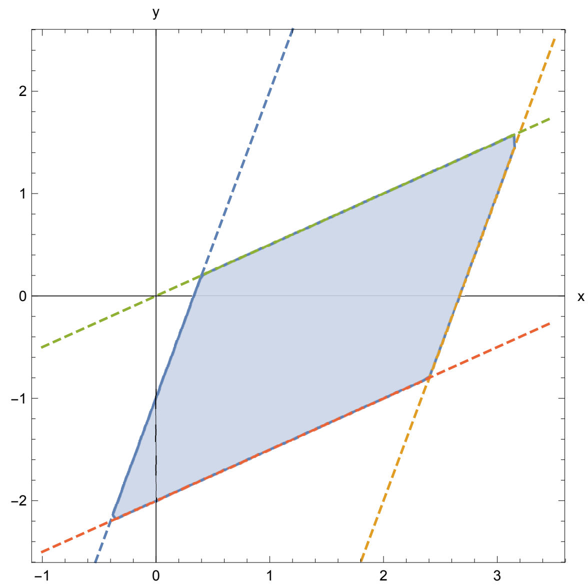
$$\int_0^a \int_0^{\pi/2} r\, d\theta\, dr + \int_a^b \int_{\arccos(a/r)}^{\pi/2} r\, d\theta\, dr + \int_b^{\sqrt{a^2+b^2}} \int_{\arccos(a/r)}^{\arcsin(b/r)} r\, d\theta\, dr$$

3. (24 points) Evaluate the integral

$$\iint_R \frac{x-2y}{3x-y}\,dA$$

where R is the parallelogram enclosed by the lines $x = 2y$, $x = 2y+4$, $y = 3x-1$ and $y = 3x-8$. Include any relevant sketches in your solution.

Solution: The region R looks like this:



Both the region R and the integrand suggest we do a substitution. From the integrand, setting

$$u = x - 2y \qquad \text{and} \qquad v = 3x - y$$

and rewriting the given lines as $x - 2y = 0$, $x - 2y = 4$, $3x - y = 1$ and $3x - y = 8$, we see that $0 \leq u \leq 4$ and $1 \leq v \leq 8$ which is a rectangular region S in the uv-plane.

Using elimination we see that

$$u - 2v = x - 6x = -5x \implies x = -\frac{1}{5}(u - 2v)$$

and

$$3u - v = -6y + y = -5y \implies y = -\frac{1}{5}(3u - v)$$

so that

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{1}{5} & \frac{2}{5} \\ -\frac{3}{5} & \frac{1}{5} \end{vmatrix} = \left(-\frac{1}{5}\right)\left(\frac{1}{5}\right) - \left(\frac{2}{5}\right)\left(-\frac{3}{5}\right) = -\frac{1}{25} + \frac{6}{25} = \frac{1}{5}$$

Finally,

$$\begin{aligned}
\iint_{R_{xy}} \frac{x-2y}{3x-y}\, dA_{xy} &= \iint_{S_{uv}} \frac{u}{v} \cdot |J(u,v)|\, dA_{uv} \\
&= \frac{1}{5} \int_0^4 \int_1^8 \frac{u}{v}\, du\, dv \\
&= \frac{1}{5} \left(\int_0^4 u\, du \right) \left(\int_1^8 \frac{1}{v}\, dv \right) \\
&= \frac{1}{5} \left[\frac{u^2}{2} \right]_0^4 \cdot \Big[\ln|v| \Big]_1^8 \\
&= \frac{1}{10} (16-0)(\ln(8) - \ln(1)) \\
&= \frac{8\ln(8)}{5}.
\end{aligned}$$

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4. (32 points) The integrals

$$V = \int_0^{2\pi} \int_0^{\pi/3} \int_0^{2\cos(\phi)} \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta + \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \int_0^1 \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$$

determine the volume of a solid object.

(a) Make a clear sketch of the cross section of the object in the rz-plane (this is a constant θ plane in cylindrical coordinates).

(b) What does the solid look like? Try to describe it in a few words of your own, technical or not, it doesn't matter. Or you can sketch a graph if you want. You shouldn't spend a lot of time on this.

(c) Express V in spherical coordinates using the order $d\phi\, d\rho\, d\theta$.

(d) Express V in cylindrical coordinates using the order $dz\, dr\, d\theta$.

(e) Evaluate any of the integrals above (including the original) to determine the value of V.

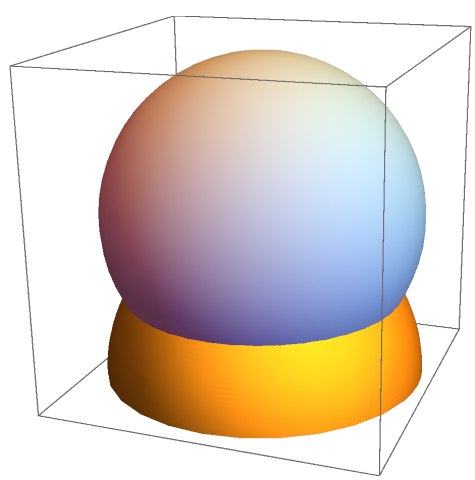
Solution: (a) The cross section in the rz-plane should look like this:

It would perhaps help to convert to Cartesian coordinates by multiplying $\rho = 2\cos(\phi)$ on both sides by ρ:

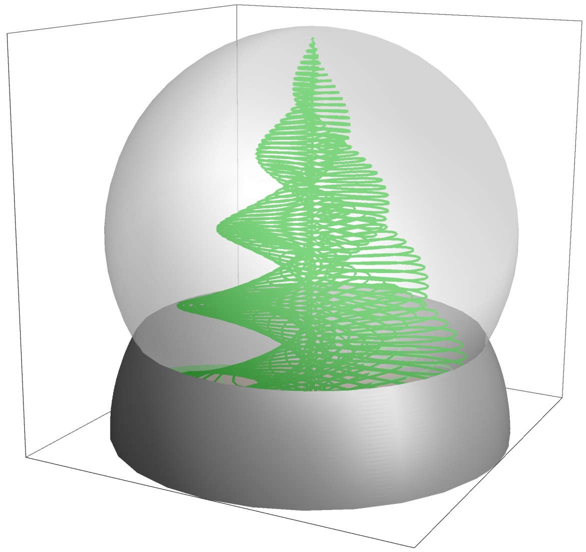
$$\rho^2 = 2\rho\cos(\phi) \implies x^2 + y^2 + z^2 = 2z \implies x^2 + y^2 + (z-1)^2 = 1$$

(complete the square).

(b) I think it looks like a snow globe, or a light fixture, or a crystal ball:

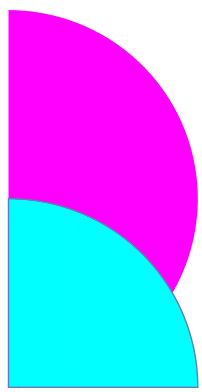


As for a snow globe, for fun check this out (yes, everything was made with functions in Mathematica. I didn't have time to figure out how make neat-looking snow inside the globe unfortunately):



Some honorable mentions from class: "ball on a pillow, light on an old police car, someone dropped a two-scoop ice cream on the ground, centerpiece, bell, alien spaceship, weird hat."

(c) Color the region like this:

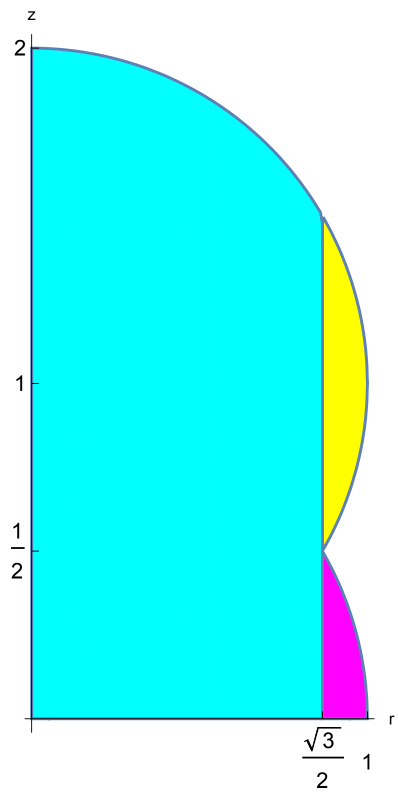


Then

$$V = \int_0^{2\pi}\int_0^{1}\int_0^{\pi/2} \rho^2 \sin(\phi)\,d\phi\,d\rho\,d\theta + \int_0^{2\pi}\int_1^{2}\int_0^{\arccos(\rho/2)} \rho^2 \sin(\phi)\,d\phi\,d\rho\,d\theta$$

Be sure to not double count the region beneath $\phi = \pi/3$ and above the lower part of $\rho = 2\cos(\phi)$.

(d) One way to do this is to color the region like this:



Convert the Cartesion coordinate equation above to cylindrical:

$$r^2 + (z-1)^2 = 1 \implies |z-1| = \sqrt{1-r^2} \implies z = 1 \pm \sqrt{1-r^2}$$

The circle of radius 1 centered at the origin in the rz-plane is, of course, $r^2 + z^2 = 1$, so we use $z = \sqrt{1-r^2}$ for the upper semicircle as usual. Then

$$V = \int_0^{2\pi} \int_0^{\sqrt{3}/2} \int_0^{1+\sqrt{1-r^2}} r\,dz\,dr\,d\theta + \int_0^{2\pi} \int_{\sqrt{3}/2}^{1} \int_0^{\sqrt{1-r^2}} r\,dz\,dr\,d\theta + \int_0^{2\pi} \int_{\sqrt{3}/2}^{1} \int_{1-\sqrt{1-r^2}}^{1+\sqrt{1-r^2}} r\,dz\,dr\,d\theta$$

(e) Just do the two given integrals, you should get:

$$\int_0^{2\pi} \int_0^{\pi/3} \int_0^{2\cos(\phi)} \rho^2 \sin(\phi)\,d\rho\,d\phi\,d\theta = \cdots = \frac{5\pi}{4}$$

and

$$\int_0^{2\pi} \int_{\pi/3}^{\pi/2} \int_0^{1} \rho^2 \sin(\phi)\,d\rho\,d\phi\,d\theta = \cdots = \frac{\pi}{3}$$

and add them together to get $V = \frac{19\pi}{12}$.

Projection, distances from a point S to a line containing a point P, and S to a plane with normal $\mathbf{n}$:

$$\text{proj}_{\mathbf{b}}\mathbf{a} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\right)\mathbf{b}, \qquad d = \frac{|\overrightarrow{PS}\times\mathbf{v}|}{|\mathbf{v}|}, \qquad d = \left|\overrightarrow{PS}\cdot\frac{\mathbf{n}}{|\mathbf{n}|}\right|$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}, \qquad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}, \qquad \mathbf{B}(t) = \mathbf{T}(t)\times\mathbf{N}(t)$$

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}, \qquad \kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}, \qquad \kappa(x) = \frac{|f''(x)|}{[1+(f'(x))^2]^{3/2}}, \qquad \frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$$

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta), \qquad \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$

$$\frac{df}{ds} = \nabla f\cdot\mathbf{u} \qquad D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - (f_{xy}(x,y))^2 \qquad \nabla f = \lambda\nabla g,\ g = 0$$

Third order Taylor approximation of $f(x,y)$ centered at (a,b):

$$\begin{aligned} f(x,y) \approx{}& f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \frac{1}{2!}\left[f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(y-b)^2\right] \\ &+ \frac{1}{3!}\left[f_{xxx}(a,b)(x-a)^3 + 3f_{xxy}(a,b)(x-a)^2(y-b) + 3f_{xyy}(a,b)(x-a)(y-b)^2 + f_{yyy}(a,b)(y-b)^3\right] \end{aligned}$$

n^{th} order Taylor approximation error centered at the point (a,b):

$$|E(x,y)| \le \frac{M}{(n+1)!}\left(|x-a| + |y-b|\right)^{n+1}$$

where M is an upper bound on the absolute value of all the relevant $n+1^{st}$ order partial derivatives of f.