APPM 2350, Summer 2018: Exam 2 June 29, 2018

INSTRUCTIONS: Please show all of your work and make your methods and reasoning clear. Answers out of the blue with no supporting work will receive no credit (unless the directions to a specific problem say otherwise, of course). No calculators or electronic devices are allowed. Please start each numbered problem on a new page in your bluebook and do the problems in order. Please sign the front of your blue book indicating you read and understood these directions in addition to the CU honor code.

SIMPLIFY ALL ANSWERS AS FAR AS POSSIBLE!!!

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1. (24 points) A few unrelated questions. Show your work in this problem.

(a) Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ or explain why it doesn't exist.

(b) Find a vector equation of the tangent line at the point $(-2, 2, 4)$ to the curve of intersection of the surface $z = 2x^2 - y^2$ and the plane $z = 4$.

Solution:

(a) This limit does not exist. Approaching the origin along the x-axis, $y = 0$ and we get

$$\lim_{(x,0)\to(0,0)} \frac{x \cdot 0^2}{x^2 + 0^4} = \lim_{x\to 0} \frac{0}{x^2} = 0$$

but if we approach the origin along the parabola $x = y^2$, then since $y \to 0$ as $y^2 \to 0$ and vice versa, we get

$$\lim_{(y^2,y)\to 0} \frac{y^2 \cdot y^2}{(y^2)^2 + y^2} = \lim_{y\to 0} \frac{y^4}{2y^4} = \lim_{y\to 0} \frac{1}{2} = \frac{1}{2}.$$

Since two different approaches to $(0, 0)$ yielded different values, $\boxed{\text{the limit does not exist}}$.

(b) The tangent line we want is in two planes: the plane $z = 4$ and the tangent *plane* to the surface $z = 2x^2 - y^2$ at the point $(-2, 2, 4)$.

The plane $z = 4$ has normal vector $\mathbf{n}_1 = \langle 0, 0, 1\rangle$.

If we write the given surface as $2x^2 - y^2 - z = 0$, then this is a level surface of the function $g(x, y, z) = 2x^2 - y^2 - z$ corresponding to $g(x, y, z) = 0$.

As such the normal vector to the tangent plane can be obtained from the gradient of g, specifically

$$\nabla g(x, y, z) = \langle 4x, -2y, -1\rangle \implies \nabla g(-2, 2, 4) = \langle -8, -4, -1\rangle = \mathbf{n}_2$$

is the normal vector to the tangent plane.

Since the tangent line we want is in both planes, to find a direction vector $\mathbf{v}$ of the line we cross the normal vectors:

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \cdots = \langle 4, -8, 0\rangle.$$

With $\mathbf{r}_0 = \langle -2, 2, 4\rangle$ the position vector of the point of tangency, the line has vector equation

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} = \langle -2, 2, 4\rangle + t\langle 4, -8, 0\rangle = \boxed{\langle -2 + 4t, 2 - 8t, 4\rangle}$$

2. (24 points) A few unrelated questions. You don't need to show any work in this problem and no partial credit will be given.

(a) TRUE or FALSE? If $f(x,y) = \cos(x) + \cos(y)$, then $-\sqrt{2} \le D_{\mathbf{u}} f(x,y) \le \sqrt{2}$ for any unit vector $\mathbf{u}$ and any point $(x,y) \in \mathbb{R}^2$.

(b) Suppose $z = f(x,y)$ where $x = g(s,t)$, $y = h(s,t)$ and you are given:

$$\begin{array}{lll} f(3,6) = 0 & f_x(3,6) = 7 & f_y(3,6) = 8 \\ g(1,2) = 3 & g_s(1,2) = -1 & g_t(1,2) = 4 \\ h(1,2) = 6 & h_s(1,2) = -5 & h_t(1,2) = 10 \end{array}$$

Find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$ when $s = 1$ and $t = 2$.

(c) Find the point(s) on the surface $x^2+4y^2-z^2 = 4$ where the tangent plane is parallel to $2x+2y+z = 5$.

Solution:

(a) $\boxed{\text{TRUE}}$. Note that $\nabla f(x,y) = \langle -\sin(x), -\sin(y)\rangle$ and for any unit vector $\mathbf{u}$ and any point (x,y), if θ is the angle between ∇f and $\mathbf{u}$, then

$$\begin{aligned} |D_{\mathbf{u}} f(x,y)| &= \Big| |\nabla f(x,y)||\mathbf{u}| \cos(\theta) \Big| \\ &= \sqrt{\sin^2(x) + \sin^2(y)} \cdot 1 \cdot |\cos(\theta)| \\ &\le \sqrt{1+1} \cdot 1 \\ &= \sqrt{2} \end{aligned}$$

since $|\cos(\theta)| \le 1$ and both $\sin^2(x)$ and $\sin^2(y)$ are always ≤ 1.

(b) Use the chain rule. Since $x = 3$ and $y = 6$ when $s = 1$ and $t = 2$, we have

$$\begin{aligned} \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \\ &= f_x g_s + f_y h_s \end{aligned}$$

so that

$$\begin{aligned} \left.\frac{\partial z}{\partial s}\right|_{(s,t)=(1,2)} &= f_x(3,6) g_s(1,2) + f_y(3,6) h_s(1,2) \\ &= 7(-1) + 8(-5) \\ &= \boxed{-47} \end{aligned}$$

Similarly,

$$\left.\frac{\partial z}{\partial t}\right|_{(s,t)=(1,2)} = 7(4) + 8(10) = \boxed{108}$$

(c) Think of the given surface as the level surface $f(x, y, z) = 4$ with $f(x, y, z) = x^2 + 4y^2 - z^2$. Then the normal vector to the tangent plane to the surface is given by $\nabla f(x, y, z) = \langle 2x, 8y, -2z\rangle$. The given plane has normal vector $\mathbf{n} = \langle 2, 2, 1\rangle$, so for the tangent plane to the surface to be parallel to the plane $2x + 2y + z = 5$ amounts to their normal vectors being parallel, that is,

$$\nabla f = \alpha \mathbf{n} \qquad \text{or} \qquad \langle 2x, 8y, -2z\rangle = \langle 2\alpha, 2\alpha, \alpha\rangle$$

for some scalar α.

This can be arranged by choosing $x = \alpha$, $y = \frac{\alpha}{4}$ and $z = -\frac{\alpha}{2}$.

But the point (x, y, z) must be on the surface too, so plug these choices into the equation of the surface:

$$\begin{aligned} \alpha^2 + 4 \cdot \left(\frac{\alpha}{4}\right)^2 - \left(-\frac{\alpha}{2}\right)^2 &= 4 \\ \alpha^2 + \frac{\alpha^2}{4} - \frac{\alpha^2}{4} &= 4 \\ \alpha^2 &= 4 \\ \alpha &= \pm 2. \end{aligned}$$

This gives two points:

$$\boxed{\left(2, \frac{1}{2}, -1\right)} \qquad \text{and} \qquad \boxed{\left(-2, -\frac{1}{2}, 1\right)}$$

3. (22 points) **Use Lagrange multipliers** to find the absolute maximum and minimum values of $f(x, y, z) = 2x + 2y + z$ subject to the constraint $x^2 + y^2 + z^2 = 9$. Show your work in this problem.

Solution: Set $g(x, y, z) = x^2 + y^2 + z^2$. We want to solve the system of equations

$$\nabla f = \lambda \nabla g \quad \text{and} \quad g(x, y, z) = 9$$

or

$$\langle 2, 2, 1 \rangle = \lambda \langle 2x, 2y, 2z \rangle \quad \text{and} \quad x^2 + y^2 + z^2 = 9$$

or

$$\begin{aligned} 2 &= 2\lambda x \\ 2 &= 2\lambda y \\ 1 &= 2\lambda z \\ x^2 + y^2 + z^2 &= 9 \end{aligned}$$

Note that λ cannot equal zero, for if it did, the first equation would yield $2 = 0$, which is false.

So solve for x, y and z in terms of λ in the first 3 equations:

$$x = \frac{1}{\lambda} \qquad y = \frac{1}{\lambda} \qquad z = \frac{1}{2\lambda}$$

and plug them into the constraint to solve for λ:

$$\begin{aligned} \left(\frac{1}{\lambda}\right)^2 + \left(\frac{1}{\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 &= 9 \\ \frac{1}{\lambda^2} + \frac{1}{\lambda^2} + \frac{1}{4\lambda^2} &= 9 \\ \frac{9}{4\lambda^2} &= 9 \\ 4\lambda^2 &= 1 \\ \lambda &= \pm\frac{1}{2}. \end{aligned}$$

Use these 2 values of λ to find x, y and z:

$$\lambda = \frac{1}{2} \implies (x, y, z) = (2, 2, 1)$$

$$\lambda = -\frac{1}{2} \implies (x, y, z) = (-2, -2, -1)$$

and note

$$f(2, 2, 1) = 4 + 4 + 1 = 9 \qquad \text{and} \qquad f(-2, -2, -1) = -4 - 4 - 1 = -9$$

So the absolute maximum value of f subject to the given constraint is $\boxed{9}$, whereas the absolute minimum value is $\boxed{-9}$.

4. (30 points) Consider the function $f(x, y) = \sin(x)\sin(y)$. Show your work in this problem.

(a) Find the critical point(s) of f that satisfy $-\pi < x < \pi$ and $-\pi < y < \pi$.

(b) Classify the critical point(s) you found in part (a) as the location of local maxima, local minima, or saddle points.

(c) At the point $(\pi/4, \pi/4)$, in what direction does f increase most rapidly?

(d) What is the maximum rate of change of f with respect to distance at the point $(\pi/4, \pi/4)$?

(e) Find the linear approximation of $f(x, y)$ at the point $(\pi/4, \pi/4)$.

(f) Use your answer from part (e) to approximate $f(\pi/4 + 0.1, \pi/4 + 0.1)$.

(g) Now, calculate an upper bound on the error associated with the linear approximation you found in part (e) that is valid for *any* values of x and y that satisfy $|x - \pi/4| \leq 0.25$ and $|y - \pi/4| \leq 0.5$.

Solution:

(a) Note ∇f always exists, so the only critical points are where $\nabla f = \mathbf{0}$. So we try to solve

$$\nabla f(x, y) = \langle \underbrace{\cos(x)\sin(y)}_{f_x}, \underbrace{\sin(x)\cos(y)}_{f_y} \rangle = \langle 0, 0 \rangle$$

Note that $f_x = 0$ when either $\cos(x) = 0 \iff x = \pm\frac{\pi}{2}$ (in the given set of x and y-values, of course) or $\sin(y) = 0 \iff y = 0$.

If $y = 0$, note $f_y(x, y) = \sin(x) \cdot 1$ which is zero only when $x = 0$, so this yields one critical point $\boxed{(0, 0)}$.

If $x = \pi/2$, then $f_y = 1 \cdot \cos(y)$ which is zero only for $y = \pm\pi/2$. So this yields two more critical points $\boxed{\left(\frac{\pi}{2}, \frac{\pi}{2}\right)}$ and $\boxed{\left(\frac{\pi}{2}, -\frac{\pi}{2}\right)}$.

If $x = -\pi/2$, then $f_y = -1 \cdot \cos(y)$, which is zero only for $y = \pm\pi/2$ again, so we get two more critical points: $\boxed{\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)}$ and $\boxed{\left(-\frac{\pi}{2}, -\frac{\pi}{2}\right)}$ for a total of five.

(b) Note that

$$\begin{aligned} f_{xx}(x, y) &= -\sin(x)\sin(y) \\ f_{xy}(x, y) &= \cos(x)\cos(y) \\ f_{yy}(x, y) &= -\sin(x)\sin(y) \end{aligned}$$

so that

$$\begin{aligned} D(x, y) &= f_{xx}f_{yy} - (f_{xy})^2 \\ &= (-\sin(x)\sin(y))(-\sin(x)\sin(y)) - (\cos(x)\cos(y))^2 \\ &= \sin^2(x)\sin^2(y) - \cos^2(x)\cos^2(y). \end{aligned}$$

Now,

$$D(0, 0) = 0 - 1 < 0 \implies \boxed{(0, 0) \text{ is a saddle point}}$$

$$D(\pi/2, \pi/2) = 1 - 0 > 0 \quad \text{and} \quad f_{xx}(\pi/2, \pi/2) = -1 < 0 \implies \boxed{(\pi/2, \pi/2) \text{ is a relative maximum}}$$

$$D(\pi/2, -\pi/2) = 1 - 0 > 0 \quad \text{and} \quad f_{xx}(\pi/2, -\pi/2) = 1 > 0 \implies \boxed{(\pi/2, -\pi/2) \text{ is a relative minimum}}$$

$$D(-\pi/2, \pi/2) = 1 - 0 > 0 \quad \text{and} \quad f_{xx}(-\pi/2, \pi/2) = 1 > 0 \implies \boxed{(-\pi/2, \pi/2) \text{ is a relative minimum}}$$

$$D(-\pi/2, -\pi/2) = 1 - 0 > 0 \quad \text{and} \quad f_{xx}(-\pi/2, -\pi/2) = -1 < 0 \implies \boxed{(-\pi/2, -\pi/2) \text{ is a relative maximum}}$$

(c) In the direction of the gradient at that point:

$$\nabla f(x, y) = \langle \cos(x)\sin(y), \sin(x)\cos(y) \rangle \implies \nabla f(\pi/4, \pi/4) = \left\langle \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \right\rangle = \boxed{\left\langle \frac{1}{2}, \frac{1}{2} \right\rangle}$$

(d) The magnitude of the gradient at that point:

$$|\nabla f(\pi/4, \pi/4)| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \boxed{\sqrt{\frac{1}{2}}}$$

(e)

$$\begin{aligned} L(x, y) &= f(\pi/4, \pi/4) + f_x(\pi/4, \pi/4)(x - \pi/4) + f_y(\pi/4, \pi/4)(y - \pi/4) \\ &= \frac{1}{2} + \frac{1}{2}(x - \pi/4) + \frac{1}{2}(y - \pi/4) \end{aligned}$$

(f)

$$\begin{aligned} f(\pi/4 + 0.1, \pi/4 + 0.1) &\approx L(\pi/4 + 0.1, \pi/4 + 0.1) \\ &= \frac{1}{2} + \frac{1}{2}(0.1) + \frac{1}{2}(0.1) \\ &= 0.5 + 0.1 \\ &= \boxed{0.6} \end{aligned}$$

(g) Note that $|f_{xx}|$, $|f_{xy}|$ and $|f_{yy}|$ are all ≤ 1 for *every* value of x and y, so we may take $M = 1$ in the error formula. Then

$$\begin{aligned} |E(x, y)| &\leq \frac{1}{2!}\left(|x - \pi/4| + |y - \pi/4|\right)^2 \\ &\leq \frac{1}{2}(0.25 + 0.5)^2 \\ &= \frac{1}{2} \cdot \frac{9}{16} \\ &= \boxed{\frac{9}{32}} \end{aligned}$$

Projection, distances from a point S to a line containing a point P, and S to a plane with normal $\mathbf{n}$:

$$\text{proj}_{\mathbf{b}}\mathbf{a} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}}\right)\mathbf{b}, \qquad d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|}, \qquad d = \left|\overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right|$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}, \qquad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}, \qquad \mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

$$\frac{d\mathbf{T}}{ds} = \kappa \mathbf{N}, \qquad \kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}, \qquad \kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}, \qquad \frac{d\mathbf{B}}{ds} = -\tau \mathbf{N}$$

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta), \qquad \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$

$$\frac{df}{ds} = \nabla f \cdot \mathbf{u} \qquad D(x,y) = f_{xx}(x,y) f_{yy}(x,y) - (f_{xy}(x,y))^2 \qquad \nabla f = \lambda \nabla g, \ g = 0$$

Third order Taylor approximation of $f(x,y)$ centered at (a,b):

$$\begin{aligned} f(x,y) \approx f(a,b) &+ f_x(a,b)(x-a) + f_y(a,b)(y-b) + f_{xx}(a,b)(x-a)^2 + f_{xy}(a,b)(x-a)(y-b) + f_{yy}(y-b)^2 \\ &+ f_{xxx}(a,b)(x-a)^3 + f_{xxy}(a,b)(x-a)^2(y-b) + f_{xyy}(a,b)(x-a)(y-b)^2 + f_{yyy}(a,b)(y-b)^3 \end{aligned}$$

n^{th} order Taylor approximation error centered at the point (a,b):

$$|E(x,y)| \leq \frac{M}{(n+1)!} \left(|x-a| + |y-b|\right)^{n+1}$$

where M is an upper bound on the absolute value of all the relevant $n+1^{st}$ order partial derivatives of f.