APPM 2350, Summer 2018: Exam 1 June 15, 2018

INSTRUCTIONS: Please show all of your work and make your methods and reasoning clear. Answers out of the blue with no supporting work will receive no credit (unless the directions to a specific problem say otherwise, of course). No calculators or electronic devices are allowed. Please start each numbered problem on a new page in your bluebook and do the problems in order. Please sign the front of your blue book indicating you read and understood these directions in addition to the CU honor code.

SIMPLIFY ALL ANSWERS AS FAR AS YOU CAN

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1. (25 points) Consider the plane $x + y + 2z = 1$.

   (a) Sketch a graph of the part of the plane that is in the first octant. Label your axes and any points of interest.

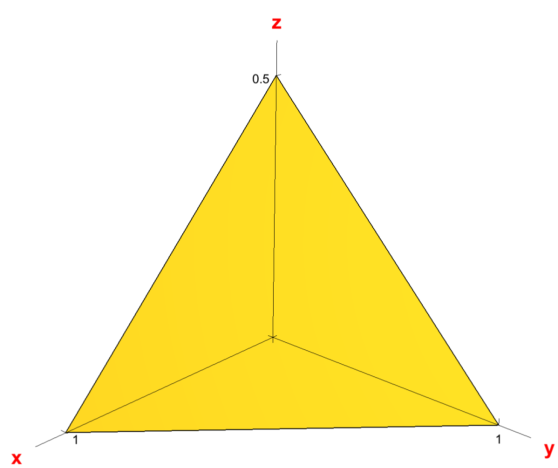
   (b) Using only Calculus 3 methods, find the area of the shape you graphed in part (a).

   (c) Find the point on the plane that is closest to the origin and say what this minimum distance is **from scratch using projections. Don't just plug stuff into a formula, show your reasoning!**.

   (d) Find an equation of the line through the origin and perpendicular to the plane.

Solution:

(a) Use the x-, y- and z-intercepts, $P_x(1,0,0)$, $P_y(0,1,0)$ and $P_z(0,0,1/2)$ respectively to graph:



(b) Choose a corner of the triangle, say at P_x, and connect this point with P_y and then P_z to get vectors $\overrightarrow{P_xP_y} = \langle -1, 1, 0\rangle$ and $\overrightarrow{P_xP_z} = \langle -1, 0, 1/2\rangle$ and the area of the triangle is

$$\frac{1}{2}\left|\overrightarrow{P_xP_y} \times \overrightarrow{P_xP_z}\right|$$

But you can easily find that

$$\overrightarrow{P_xP_y} \times \overrightarrow{P_xP_z} = \langle 1/2, 1/2, 1\rangle$$

so that

$$\left|\overrightarrow{P_xP_y} \times \overrightarrow{P_xP_z}\right| = \sqrt{(1/2)^2 + (1/2)^2 + 1^2} = \sqrt{1/4 + 1/4 + 1} = \sqrt{3/2}$$

and so the area of the triangle is

$$\boxed{\frac{1}{2}\sqrt{\frac{3}{2}}}$$

(c) Connect the origin to a point you know is on the plane, let's use P_x because it's easy, and form the position vector $\mathbf{v} = \langle 1, 0, 0\rangle$. All we need to do is project $\mathbf{v}$ onto the normal vector of the plane, which is $\mathbf{n} = \langle 1, 1, 2\rangle$, and then take the magnitude of this projection vector.

So

$$\text{proj}_{\mathbf{n}}\mathbf{v} = \left(\frac{\mathbf{v}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\right)\mathbf{n} = \left(\frac{1+0+0}{1+1+2^2}\right)\langle 1^2, 1^2, 2\rangle = \langle 1/6, 1/6, 2/6\rangle$$

So the point we seek is $\boxed{(1/6, 1/6, 1/3)}$ and the minimum distance is

$$|\text{proj}_{\mathbf{n}}\mathbf{v}| = \sqrt{(1/6)^2 + (1/6)^2 + (2/6)^2} = \sqrt{6/36} = \boxed{1/\sqrt{6}}$$

(d) Use $\mathbf{r}(t) = \langle 0, 0, 0\rangle + t\mathbf{n}$ since we know the points $(0, 0, 0)$ is on the line and if the line is perpendicular to the plane, then we can use the normal vector $\mathbf{n} = \langle 1, 1, 2\rangle$ of the plane as the direction vector of the line. This simplifies to

$$\boxed{\mathbf{r}(t) = \langle t, t, 2t\rangle}$$

2. (24 points) A few unrelated questions. Just provide answers for this problem – you don't have to show any work and no partial credit will be given.

(a) TRUE or FALSE? If $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$ are orthogonal, then the vectors $\mathbf{a}$ and $\mathbf{b}$ must have the same length.

(b) Classify the quadric surface $x^2+y^2+z^2-2x-4y-8z=15$ as completely as you can.

(c) Consider the surfaces $z=x^2$ and $x^2+y^2=1$.

  (i) Describe both surfaces using two words each.

  (ii) Find a vector-valued function that represents the curve of intersection of the two surfaces using basic trig functions.

Solution:

(a) It's $\boxed{\text{TRUE}}$. You should draw a picture. But a formal proof would go like this: We are given that $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})=0$ since they are orthogonal. But we can simplify this using properties of the dot product:

$$\begin{aligned}
(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) &= 0 \\
\mathbf{a}\cdot(\mathbf{a}-\mathbf{b})+\mathbf{b}\cdot(\mathbf{a}-\mathbf{b}) &= 0 \\
\mathbf{a}\cdot\mathbf{a}\underbrace{-\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{a}}_{\text{these cancel}}-\mathbf{b}\cdot\mathbf{b} &= 0 \\
|\mathbf{a}|^2-|\mathbf{b}|^2 &= 0 \\
|\mathbf{a}|^2 &= |\mathbf{b}|^2 \\
\sqrt{|\mathbf{a}|^2} &= \sqrt{|\mathbf{b}|^2} \\
\Big||\mathbf{a}|\Big| &= \Big||\mathbf{b}|\Big| \\
|\mathbf{a}| &= |\mathbf{b}|
\end{aligned}$$

since the absolute values are unnecessary because magnitudes of vectors are nonnegative always.

(b) Rearrange and complete the square:

$$\begin{aligned}
x^2+y^2+z^2-2x-4y-8z &= 15 \\
x^2-2x+1-1+y^2-4y+4-4+z^2-8z+16-16 &= 15 \\
(x-1)^2-1+(y-2)^2-4+(z-4)^2-16 &= 15 \\
(x-1)^2+(y-2)^2+(z-4)^2 &= 36
\end{aligned}$$

which is a $\boxed{\text{sphere of radius 6 centered at } (1,2,4)}$

(c) (i) $\boxed{z=x^2 \text{ is a parabolic cylinder}}$ whereas $\boxed{x^2+y^2=1 \text{ is a circular cylinder}}$

(ii) Set $x(t)=\cos(t)$, $y(t)=\sin(t)$ so that $z(t)=\cos^2(t)$. We get

$$\boxed{\mathbf{r}(t)=\langle\cos(t),\sin(t),\cos^2(t)\rangle}$$

3. (25 points) Consider the function $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), e^t \rangle$ on the interval $0 \le t \le \pi$.

(a) Describe the given curve in one sentence as specifically as you can.

(b) Find the point on the curve where the tangent line is parallel to the plane $\sqrt{3}x + y = 1$.

(c) Set up a simplified integral that represents the arc length of the given curve. **You do not have to actually evaluate this integral!**

Solution:

(a) The curve is a portion of a helix, on a circular cylinder of radius 2 elongated along the z-axis and rotating counterclockwise, whose z-coordinate increases from 1 to e^{π} on the given interval.

(b) The tangent line has direction vector

$$\mathbf{r}'(t) = \langle -2\sin(t), 2\cos(t), e^t \rangle$$

and for the tangent line to be parallel to the plane, the line's direction vector has to be orthogonal to the normal vector of the plane $\mathbf{n} = \langle \sqrt{3}, 1, 0 \rangle$. So we solve

$$\begin{aligned} \mathbf{r}'(t) \cdot \mathbf{n} &= 0 \\ -2\sqrt{3}\sin(t) + 2\cos(t) + 0 &= 0 \\ 2\sqrt{3}\sin(t) &= 2\cos(t) & \qquad t = \frac{\pi}{2} \text{ is not the solution so divide by } \cos(t) \\ \tan(t) &= \frac{1}{\sqrt{3}} \\ t &= \frac{\pi}{6} \end{aligned}$$

since $0 \le t \le \pi$ is given.

The corresponding point we want has position vector

$$\mathbf{r}(\pi/6) = \langle 2\cos(\pi/6), 2\sin(\pi/6), e^{\pi/6} \rangle = \langle 2(\sqrt{3}/2), 2(1/2), e^{\pi/6} \rangle = \boxed{\langle \sqrt{3}, 1, e^{\pi/6} \rangle}$$

or you can also say the actual point we want is

$$\boxed{(\sqrt{3}, 1, e^{\pi/6})}$$

(c) The arc length is

$$\int_0^{\pi} |\mathbf{r}'(t)|\, dt$$

so we find

$$\begin{aligned} |\mathbf{r}'(t)| &= \sqrt{[-2\sin(t)]^2 + [2\cos(t)]^2 + (e^t)^2} \\ &= \sqrt{4[\sin^2(t) + \cos^2(t)] + e^{2t}} \\ &= \sqrt{4 \cdot 1 + e^{2t}} \end{aligned}$$

so the integral we want is

$$\boxed{\int_0^{\pi} \sqrt{4 + e^{2t}}\, dt}$$

4. (26 points) Consider a particle with position function $\mathbf{r}(t) = \langle t\cos(t), t\sin(t), t\rangle$.

(a) Find the velocity of the particle when $t = 0$.

(b) Find the speed versus time function $v(t)$.

(c) Find the speed of the particle when $t = 0$.

(d) Find the unit tangent vector $\mathbf{T}(0)$.

(e) Find the acceleration vector $\mathbf{a}(0)$.

(f) Find the curvature κ when $t = 0$.

(g) Now, I assume you don't want to calculate $\mathbf{N}(0)$ directly from its definition. Don't worry, you don't have to do that, so please don't. However, I wonder if you could find $\mathbf{N}(0)$ indirectly using what you've already done in parts (a) through (f) of this problem and a little bit of work. What do you think? Could you? Simply answer YES or NO.

Solution:

(a)

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle \cos(t) - t\sin(t), \sin(t) + t\cos(t), 1\rangle$$

so

$$\mathbf{v}(0) = \langle 1 - 0, 0 + 0, 1\rangle = \boxed{\langle 1, 0, 1\rangle}$$

(b)

$$\begin{aligned} v(t) &= |\mathbf{v}(t)| \\ &= \sqrt{[\cos(t) - t\sin(t)]^2 + [\sin(t) + t\cos(t)]^2 + 1^2} \\ &= \sqrt{\cos^2(t) - 2t\sin(t)\cos(t) + t^2\sin^2(t) + \sin^2(t) + 2t\sin(t)\cos(t) + t^2\cos^2(t) + 1^2} \\ &= \sqrt{\sin^2(t) + \cos^2(t) + t^2[\sin^2(t) + \cos^2(t)] + 1} \\ &= \boxed{\sqrt{2 + t^2}} \end{aligned}$$

(c) $v(0) = \sqrt{2 + 0^2} = \boxed{\sqrt{2}}$

(d)

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \frac{1}{\sqrt{2}}\langle 1, 0, 1\rangle = \boxed{\langle 1/\sqrt{2}, 0, 1/\sqrt{2}\rangle}$$

(e)

$$\begin{aligned} \mathbf{a}(t) &= \mathbf{r}''(t) \\ &= \langle -\sin(t) - [\sin(t) + t\cos(t)], \cos(t) + \cos(t) - t\sin(t), 0\rangle \\ &= \langle -2\sin(t) - t\cos(t), 2\cos(t) - t\sin(t), 0\rangle \end{aligned}$$

so

$$\mathbf{a}(0) = \langle 0 - 0, 2 \cdot 1 - 0, 0\rangle = \boxed{\langle 0, 2, 0\rangle}$$

(f) Use

$$\kappa(0) = \frac{|\mathbf{r}'(0) \times \mathbf{r}''(0)|}{|\mathbf{r}'(0)|^3}$$

But

$$\mathbf{r}'(0) \times \mathbf{r}''(0) = \langle -2, 0, 2 \rangle \implies |\mathbf{r}'(0) \times \mathbf{r}''(0)| = \sqrt{(-2)^2 + 2^2} = \sqrt{8}$$

and

$$|\mathbf{r}'(0)|^3 = (\sqrt{2})^3 = \sqrt{8}$$

so that

$$\kappa(0) = \frac{\sqrt{8}}{\sqrt{8}} = \boxed{1}.$$

(g) YES. Recall that

$$\mathbf{a}(t) = \underbrace{v'(t)}_{a_T}\,\mathbf{T}(t) + \underbrace{\kappa(t)[v(t)]^2}_{a_N}\,\mathbf{N}(t)$$

so use part (b) to easily find $v'(t)$ and then $v'(0)$. Everything else in the above equation is known to you at $t = 0$ except $\mathbf{N}(0)$, so plug in $t = 0$ and everything else you've just calculated and solve for $\mathbf{N}(0)$.

Projection, distances from a point S to a line containing a point P, and S to a plane with normal $\mathbf{n}$:

$$\text{proj}_{\mathbf{b}}\mathbf{a} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}}\right)\mathbf{b}, \qquad d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|}, \qquad d = \left|\overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right|$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}, \qquad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}, \qquad \mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}, \qquad \kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}, \qquad \kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}, \qquad \frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$$

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta), \qquad \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$