1. [2350/050724 (16 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given. Please write your answers in a single column separate from any work you do to arrive at the answer.

   (a) Given any three nonzero vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$, if $\mathbf{u} \times \mathbf{v} \times \mathbf{w} = \mathbf{0}$, then the vectors must always all lie in the same plane.

   (b) The torsion, $\tau = \dfrac{[\mathbf{r}'(t) \times \mathbf{r}''(t)] \cdot \mathbf{r}'''(t)}{||\mathbf{r}'(t) \times \mathbf{r}''(t)||^2}$, of the curve $\mathbf{r}(t) = \langle \sin t, 2, \cos t\rangle$ is $\tau = 2$.

   (c) The line with symmetric equations $x = -0.5y = z$ never intersects the plane $x + y + z = 1$

   (d) $\dfrac{1}{4}x^2 - 2x + y^2 - 10y - z^2 + 39 = 0$ is a hyperboloid of two sheets.

   (e) For any vectors $\mathbf{A}, \mathbf{B}$, the operation $\nabla \times [\nabla \times (\mathbf{A} \times \mathbf{B})]$ is well-defined.

   (f) The direction of motion of a particle moving on the path $\mathbf{r}(t) = (1 - 3e^t)\,\mathbf{i} + (2 - 2e^t)\,\mathbf{j} + (3 - e^t)\,\mathbf{k}, t \in \mathbb{R}$, is always changing (the normal component of the acceleration is never 0) but its speed is constant (its tangential acceleration is 0).

   (g) If $g(x, y) \to 1$ when $(x, y) \to (0, 0)$ along the y-axis and $g(x, y) \to 1$ when $(x, y) \to (0, 0)$ along the x-axis, and $g(0, 0) = 1$, then $g(x, y)$ must be continuous at $(0, 0)$.

   (h) The linear Taylor polynomial of $f(x, y) = e^{-x^4-y^4}$ centered at $(1, 1)$ is $T_1(x, y) = e^{-2}(9 - 4x - 4y)$.

   SOLUTION:

   (a) **FALSE** Consider $\mathbf{i} \times \mathbf{j} \times \mathbf{k}$.

   (b) **FALSE** Since the curve lies in a plane, its torsion is 0.

$$\mathbf{r}'(t) = \langle \cos t, 0, -\sin t\rangle \qquad \mathbf{r}''(t) = \langle -\sin t, 0, -\cos t\rangle \qquad \mathbf{r}'''(t) = \langle -\cos t, 0, \sin t\rangle$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & 0 & -\sin t \\ -\sin t & 0 & -\cos t \end{vmatrix} = \mathbf{j}$$

$$\tau = \frac{[\mathbf{r}'(t) \times \mathbf{r}''(t)] \cdot \mathbf{r}'''(t)}{||\mathbf{r}'(t) \times \mathbf{r}''(t)||^2} = \frac{\langle 0, 1, 0\rangle \cdot \langle -\cos t, 0, \sin t\rangle}{\|\mathbf{j}\|^2} = \frac{0}{1} = 0$$

   (c) **TRUE** Alternative 1: Substituting the equation of the line into the equation of the plane yields $-0.5y + y - 0.5y = 0 \neq 1$. Alternative 2: The direction vector of the line is $\langle 1, -2, 1\rangle$ and the normal vector to the plane is $\langle 1, 1, 1\rangle$. The dot product of these two vectors is zero, implying that the line is parallel to the plane and thus never intersects it.

   (d) **TRUE** Complete the square:

$$\frac{1}{4}\left(x^2 - 8x + 16 - 16\right) + y^2 - 10y + 25 - 25 - z^2 + 39 = 0 \implies \left(\frac{x-4}{2}\right)^2 + (y-5)^2 - z^2 = -10$$

   (e) **TRUE** $\mathbf{A} \times \mathbf{B}$ is a vector. $\nabla \times (\mathbf{A} \times \mathbf{B})$ yields another vector which possesses a curl, $\nabla \times [\nabla \times (\mathbf{A} \times \mathbf{B})]$.

   (f) **FALSE** The path is a straight line, $\mathbf{r}(t) = \langle 1, 2, 3\rangle + e^t\langle -3, -2, -1\rangle$, which has no curvature ($\kappa = 0$) and thus particles never change direction $\left(a_N = \kappa\|\mathbf{v}\|^2 = 0\right)$. The speed on the path is

$$\|\mathbf{v}\| = \|\mathbf{r}'(t)\| = \sqrt{\left(-3e^t\right)^2 + \left(-2e^t\right)^2 + \left(-e^t\right)^2} = \sqrt{14}e^t \implies a_T = \frac{\mathrm{d}\|\mathbf{v}\|}{\mathrm{d}t} = \sqrt{14}e^t$$

   which is always positive, implying that the particle's speed is always increasing.

   (g) **FALSE** The fact that $g(x, y) \to 1$ as $(x, y) \to (0, 0)$ along two different paths does not imply that $\lim\limits_{(x,y)\to(0,0)} g(x, y) = 1$ (or that the limit even exists) and thus we cannot guarantee anything about the continuity of $g(x, y)$ at $(0, 0)$.

   (h) **TRUE**

$$f_x(x, y) = -4x^3e^{-x^4-y^4} \implies f_x(1, 1) = -4e^{-2} \qquad f_y(x, y) = -4y^3e^{-x^4-y^4} \implies f_y(1, 1) = -4e^{-2}$$

$$T_1(x, y) = e^{-2} - 4e^{-2}(x - 1) - 4e^{-2}(y - 1) = e^{-2}\left(9 - 4x - 4y\right)$$

2. [2350/050724 (30 pts)] Consider the triangular region, $\mathcal{D}$, with boundary, $\partial\mathcal{D}$, given by $x = 1, y = 0, y = x$, and the vector field $\mathbf{F} = 3(x-1)^2 y\,\mathbf{i} + (x-1)y^3\,\mathbf{j}$.

(a) (15 pts) Without using any Calculus 3 theorems, directly compute the circulation of $\mathbf{F}$ along $\partial\mathcal{D}$ with *clockwise* orientation.

(b) (15 pts) Use Green's Theorem to compute the outward flux of $\mathbf{F}$ through $\partial\mathcal{D}$.

SOLUTION:

(a) The boundary, $\partial\mathcal{D}$, is piecewise smooth so computing the circulation requires three line integrals. However, the vector field vanishes on $x = 1$ and $y = 0$ so we only need to compute the line integral along $y = x$, which we denote as $\mathcal{C}$. We parameterize this as $x = t, y = t, 0 \le t \le 1$. Then $\mathrm{d}x = \mathrm{d}y = \mathrm{d}t$ and

$$\text{Circulation} = \oint_{\partial\mathcal{D}} 3(x-1)^2 y\,\mathrm{d}x + (x-1)y^3\,\mathrm{d}y = \int_{\mathcal{C}} 3(x-1)^2 y\,\mathrm{d}x + (x-1)y^3\,\mathrm{d}y$$

$$= \int_0^1 \left[3(t-1)^2 t + (t-1)t^3\right]\mathrm{d}t = \int_0^1 \left[3t\left(t^2 - 2t + 1\right) + t^4 - t^3\right]\mathrm{d}t$$

$$= \int_0^1 \left(t^4 + 2t^3 - 6t^2 + 3t\right)\mathrm{d}t = \left(\frac{t^5}{5} + \frac{t^4}{2} - 2t^3 + \frac{3t^2}{2}\right)\Bigg|_0^1 = \frac{1}{5} + \frac{1}{2} - 2 + \frac{3}{2} = \frac{1}{5}$$

Alternatively, $\mathbf{r}(t) = \langle t, t\rangle$, $0 \le t \le 1, \mathbf{r}'(t) = \langle 1, 1\rangle$, $\mathbf{F}[\mathbf{r}(t)] = \left\langle 3t(t-1)^2, t^3(t-1)\right\rangle$, $\mathbf{F}[\mathbf{r}(t)] \cdot \mathbf{r}'(t) = t^4 + 2t^3 - 6t^2 + 3t$ and

$$\text{Circulation} = \oint_{\partial\mathcal{D}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_0^1 \left(t^4 + 2t^3 - 6t^2 + 3t\right)\mathrm{d}t = \frac{1}{5}$$

(b) Using Green's Theorem we have

$$\text{Flux} = \oint_{\partial\mathcal{D}} \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}s = \iint_{\mathcal{D}} \nabla \cdot \mathbf{F}\,\mathrm{d}A$$

$$= \int_0^1 \int_0^x \left[6(x-1)y + 3(x-1)y^2\right]\mathrm{d}y\,\mathrm{d}x = \int_0^1 \int_0^x (x-1)\left(6y + 3y^2\right)\mathrm{d}y\,\mathrm{d}x$$

$$= \int_0^1 (x-1)\left[\left(3y^2 + y^3\right)\right]\Big|_0^x \mathrm{d}x = \int_0^1 \left[(x-1)\left(3x^2 + x^3\right)\right]\mathrm{d}x$$

$$= \int_0^1 \left(x^4 + 2x^3 + -3x^2\right)\mathrm{d}x = \left(\frac{x^5}{5} + \frac{x^4}{2} - x^3\right)\Bigg|_0^1 = \left(\frac{1}{5} + \frac{1}{2} - 1\right) = -\frac{3}{10}$$

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3. [2350/050724 (30 pts)] Consider the function $f(x, y) = x^4 + y^4 - 2x^2 - 4y$

(a) (11 pts) Suppose you are standing on the surface with $x = 2, y = -1$.

i. (2 pts) What is the distance to the xy-plane from your position?

ii. (2 pts) Is the xy-plane above or below you?

iii. (7 pts) Find a unit vector in the xy-plane showing the direction you would need to move to follow the level curve that passes through your x, y coordinates.

(b) (7 pts) Suppose you are walking on the surface along a path whose projection on the xy-plane is $\mathbf{r}(t) = \left(4t - \frac{3}{2}\right)\mathbf{i} + 2t\,\mathbf{j}$. Find the rate of change with respect to time of your z-coordinate when your path passes through the point $(x, y, z) = \left(-\frac{1}{2}, \frac{1}{2}, -\frac{19}{8}\right)$.

(c) (12 pts) Find and classify all critical points of the function.

SOLUTION:

(a) i. $f(2, -1) = 2^4 + (-1)^4 - 2(2)^2 - 4(-1) = 13$

ii. Since $13 > 0$ the xy-plane is below you.

iii. Following a level curve means the rate of change of $f(x, y)$ with respect to distance, the directional derivative, is 0. Let $\mathbf{u} = \langle u_1, u_2\rangle$ be the unit vector we seek.

$$f_x = 4x^3 - 4x \qquad f_y = 4y^3 - 4$$

$$\frac{\mathrm{d}f}{\mathrm{d}s}\bigg|_{(2,-1)} = D_{\mathbf{u}} f(2, -1) = \nabla f(2, -1) \cdot \mathbf{u} = f_x(2, -1)u_1 + f_y(2, -1)u_2 = 24u_1 - 8u_2 = 0 \implies 3u_1 = u_2$$

With $u_1 = 1, u_2 = 3$. A unit vector that will satisfy the given condition is $\mathbf{u} = \pm\frac{1}{\sqrt{10}}\langle 1,3\rangle$

(b) You arrive at the point in question when $t = \frac{1}{4}$. We also have $\mathbf{r}'(t) = \langle 4,2\rangle$. Using the chain rule,

$$\left.\frac{\mathrm{d}z}{\mathrm{d}t}\right|_{t=\frac{1}{4}} = \nabla f(-\tfrac{1}{2},\tfrac{1}{2})\cdot\mathbf{r}'\left(\tfrac{1}{4}\right) = \left\langle 4\left(-\tfrac{1}{2}\right)^3 - 4\left(-\tfrac{1}{2}\right), 4\left(\tfrac{1}{2}\right)^3 - 4\right\rangle\cdot\langle 4,2\rangle = \left\langle \tfrac{3}{2}, -\tfrac{7}{2}\right\rangle\cdot\langle 4,2\rangle = -1$$

(c) Using the gradient equations from part (a)iii, we have

$$4x^3 - 4x = 4x\left(x^2-1\right) = 4x(x+1)(x-1) = 0 \implies x = -1,0,1$$
$$4y^3 - 4 = 0 \implies y = 1$$

So the critical points are $(-1,1),(0,1),(1,1)$. To classify them we use the Second Derivatives Test.

$$f_{xx} = 12x^2 - 4 \qquad f_{xy} = 0 \qquad f_{yy} = 12y^2 \qquad D(x,y) = 144x^2y^2 - 48y^2$$
$$D(-1,1) = 144 - 48 = 96 > 0,\ f_{yy}(-1,1) = 12 > 0 \implies f(-1,1) \text{ is a local minimum}$$
$$D(0,1) = -48 < 0 \implies (0,1) \text{ is a saddle point}$$
$$D(1,1) = 144 - 48 = 96 > 0,\ f_{yy}(1,1) = 12 > 0 \implies f(1,1) \text{ is a local minimum}$$

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4. [2350/050724 (20 pts)] A butterfly of the species *Infinitus spectacularis* has been tagged with a radio receiver that measures the amount of work it does when flying around the first octant. The vector field in which the butterfly is flying is

$$\mathbf{F} = \left[ye^{xy}\ln(yz) - z\sin(xz)\right]\mathbf{i} + \left[\frac{e^{xy}}{y} + xe^{xy}\ln(yz)\right]\mathbf{j} + \left[\frac{e^{xy}}{z} - x\sin(xz) + 3z^2\right]\mathbf{k}$$

Biologist records over time indicate that the work done by the butterfly for every closed path it flies is always zero. How much work is done by the butterfly if it flies along the straight line path from $(1,1,1)$ to $\left(\frac{1}{2},4,2\right)$?

SOLUTION:

Since the work done on every closed path is zero, the vector field is conservative, so a potential function f exists such that $\mathbf{F} = \nabla f$.

$$\frac{\partial f}{\partial x} = ye^{xy}\ln(yz) - z\sin(xz) \implies f(x,y,z) = \int\left[ye^{xy}\ln(yz) - z\sin(xz)\right]\mathrm{d}x = e^{xy}\ln(yz) + \cos(xz) + g(y,z)$$
$$\frac{\partial f}{\partial y} = \frac{e^{xy}}{y} + xe^{xy}\ln(yz) + \frac{\partial g}{\partial y} = \frac{e^{xy}}{y} + xe^{xy}\ln(yz) \implies \frac{\partial g}{\partial y} = 0 \implies g(y,z) = h(z)$$
$$\implies f(x,y,z) = e^{xy}\ln(yz) + \cos(xz) + h(z)$$
$$\frac{\partial f}{\partial z} = \frac{e^{xy}}{z} - x\sin(xz) + \frac{\mathrm{d}h}{\mathrm{d}z} = \frac{e^{xy}}{z} - x\sin(xz) + 3z^2 \implies \frac{\mathrm{d}h}{\mathrm{d}z} = 3z^2 \implies h(z) = z^3 + c$$
$$f(x,y,z) = e^{xy}\ln(yz) + \cos(xz) + z^3 + c$$

Now use the fundamental theorem of line integrals to find the work

$$\text{Work} = \int_{\mathcal{C}}\mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_{(1,1,1)}^{\left(\frac{1}{2},4,2\right)}\nabla f\cdot\mathrm{d}\mathbf{r} = f\left(\frac{1}{2},4,2\right) - f(1,1,1)$$
$$= e^2\ln 8 + \cos 1 + 8 - (e\ln 1 + \cos 1 + 1) = e^2\ln 8 + 7 = 3e^2\ln 2 + 7$$

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5. [2350/050724 (34 pts)] Let $\mathcal{W}$ be the solid bounded by the planes $x=0, y=0, z=2$ and the portion of $2z = x^2+y^2$ above the fourth quadrant, and let $\partial\mathcal{W}$ denote its boundary. We will be considering the vector field $\mathbf{E} = \left\langle x^2y, xy^2, xy(z-2)\right\rangle$. The identity $\sin 2x = 2\sin x\cos x$ might be helpful.

(a) (4 pts) Briefly explain why computing the flux of $\mathbf{E}$ through $\partial\mathcal{W}$ requires evaluating only a single nontrivial integral.

(b) (15 pts) Find the outward flux of $\mathbf{E}$ through $\partial\mathcal{W}$ by direct calculation.

(c) (15 pts) Use an important Calculus 3 theorem to compute the outward flux of $\mathbf{E}$ through $\partial\mathcal{W}$ another way.

SOLUTION:

(a) $\mathbf{E} = \mathbf{0}$ on the planes, $x = 0, y = 0$. On the plane $z = 2$, the vector field has no $\mathbf{k}$-component. In all cases $\mathbf{E} \cdot \mathbf{n} = 0$ and thus there is no flux through those planes. Consequently, we only need to compute the flux through the paraboloid via a nontrivial integral to find the flux through all of $\partial\mathcal{W}$.

(b) We find the flux through the portion of the circular paraboloid, $\mathcal{S}$, by projecting it onto the xy-plane. Then the region of integration, $\mathcal{R}$, is the fourth quadrant portion of the disk of radius 2, $\mathbf{p} = \mathbf{k}$, $g(x, y, z) = x^2 + y^2 - 2z$, $\nabla g = \langle 2x, 2y, -2\rangle$ and $|\nabla g \cdot \mathbf{p}| = 2$. Outward flux corresponds to a downward pointing vector so we use $+\nabla g$.

$$\begin{aligned}
\mathbf{E} \cdot \frac{\nabla g}{|\nabla g \cdot \mathbf{p}|} &= \left\langle x^2 y, xy^2, xy(z-2)\right\rangle \cdot \frac{\langle 2x, 2y, -2\rangle}{2} \\
&= \frac{1}{2}\left[2x^3 y + 2xy^3 - 2xy(z-2)\right] \\
&= xy\left(x^2 + y^2\right) - xy(z-2) \qquad \text{(eliminate } z \text{ using the surface)} \\
&= xy\left(x^2 + y^2\right) - xy\left[\frac{1}{2}\left(x^2 + y^2\right) - 2\right] \\
&= \frac{1}{2}xy\left(x^2 + y^2\right) + 2xy = \frac{1}{2}xy\left[\left(x^2 + y^2\right) + 4\right]
\end{aligned}$$

Since the flux through the planes is zero, we have

$$\begin{aligned}
\text{Flux} &= \iint_{\partial\mathcal{W}} \mathbf{E} \cdot \mathbf{n}\, \mathrm{d}S = \iint_{\mathcal{S}} \mathbf{E} \cdot \mathbf{n}\, \mathrm{d}S = \iint_{\mathcal{R}} \frac{1}{2}xy\left[\left(x^2 + y^2\right) + 4\right] \mathrm{d}A \qquad \text{(switch to polar coordinates)} \\
&= \frac{1}{2}\int_{3\pi/2}^{2\pi}\int_0^2 (r\sin\theta)(r\cos\theta)\left(r^2 + 4\right) r\, \mathrm{d}r\, \mathrm{d}\theta \\
&= \frac{1}{2}\left(\frac{1}{2}\int_{3\pi/2}^{2\pi} \sin 2\theta\right)\left[\int_0^2 \left(r^5 + 4r^3\right) \mathrm{d}r\right] \\
&= \frac{1}{2}\left[-\frac{1}{4}\cos 2\theta\Big|_{3\pi/2}^{2\pi}\left(\frac{r^6}{6} + r^4\right)\Big|_0^2\right] = -\frac{1}{8}\left[1 - (-1)\right]\left[\frac{2^6}{6} + 16\right] \\
&= -\frac{1}{4}\left(\frac{32}{3} + \frac{48}{3}\right) = -\frac{20}{3}
\end{aligned}$$

(c) Use Gauss' Divergence theorem.

$$\begin{aligned}
\text{Flux} &= \iiint_{\mathcal{W}} \nabla \cdot \mathbf{E}\, \mathrm{d}V \\
&= \iiint_{\mathcal{W}} (2xy + 2xy + xy)\, \mathrm{d}V = \iiint_{\mathcal{W}} 5xy\, \mathrm{d}V \qquad \text{(use cylindrical coordinates)} \\
&= \int_{3\pi/2}^{2\pi}\int_0^2\int_{r^2/2}^2 5(r\cos\theta)(r\sin\theta)\, r\, \mathrm{d}z\, \mathrm{d}r\, \mathrm{d}\theta = \frac{5}{2}\int_{3\pi/2}^{2\pi}\int_0^2\int_{r^2/2}^2 r^3 \sin 2\theta\, \mathrm{d}z\, \mathrm{d}r\, \mathrm{d}\theta \\
&= \frac{5}{2}\int_{3\pi/2}^{2\pi}\int_0^2 r^3 \sin 2\theta\, z\Big|_{r^2/2}^2 \mathrm{d}r\, \mathrm{d}\theta = \frac{5}{2}\int_{3\pi/2}^{2\pi}\int_0^2 r^3 \sin 2\theta\left(2 - \frac{r^2}{2}\right) \mathrm{d}r\, \mathrm{d}\theta \\
&= \frac{5}{2}\left(\int_{3\pi/2}^{2\pi} \sin 2\theta\, \mathrm{d}\theta\right)\left(\frac{r^4}{2} - \frac{r^6}{12}\right)\Big|_0^2 \\
&= -\frac{5}{4}\cos 2\theta\Big|_{3\pi/2}^{2\pi}\left(8 - \frac{16}{3}\right) \\
&= -\frac{5}{4}\left[1 - (-1)\right]\left(\frac{8}{3}\right) = -\frac{20}{3}
\end{aligned}$$

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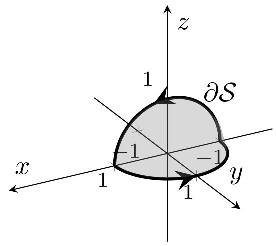
6. [2350/050724 (20 pts)] Use Stokes theorem to evaluate $\int_{\partial\mathcal{S}} x\,\mathrm{d}x + (x-2yz)\,\mathrm{d}y + \left(x^2+z^4\right)\mathrm{d}z$. $\partial\mathcal{S}$ and its orientation is shown in the figure and consists of two semicircles, $x^2+y^2=1$ and $x^2+z^2=1$, lying on the unit sphere. A portion of the sphere is shown as the shaded region.



SOLUTION:

Let $\mathcal{S}$ be the portion of the unit sphere whose boundary is $\partial\mathcal{S}$. Then we need to compute $\iint_{\mathcal{S}} \nabla\times\mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S$ where $\mathbf{F} = \langle x, x-2yz, x^2+z^4\rangle$.

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & x-2yz & x^2+z^4 \end{vmatrix} = 2y\,\mathbf{i} - 2x\,\mathbf{j} + \mathbf{k}$$

The surface is $g(x,y,z) = x^2+y^2+z^2 \implies \nabla g = \langle 2x, 2y, 2z\rangle$. Project onto the xy-plane so that $\mathbf{p}=\mathbf{k} \implies |\nabla g\cdot\mathbf{p}| = 2z$ since $z\geq 0$. The region of integration, $\mathcal{R}$, is the portion of the unit disk where $y\geq 0$. Given the orientation of the boundary of the surface, we choose $+\nabla g$. Then

$$\nabla\times\mathbf{F}\cdot\frac{+\nabla g}{|\nabla g\cdot\mathbf{p}|} = \langle 2y, -2x, 1\rangle\cdot\frac{\langle 2x, 2y, 2z\rangle}{2z} = 1$$

Thus,

$$\int_{\partial\mathcal{S}} x\,\mathrm{d}x + (x-2yz)\,\mathrm{d}y + \left(x^2+z^4\right)\mathrm{d}z = \iint_{\mathcal{S}} \nabla\times\mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S = \iint_{\mathcal{R}} 1\,\mathrm{d}A = \iint_{\mathcal{R}} \mathrm{d}A = \text{area}(\mathcal{R}) = \frac{\pi}{2}$$

Alternatively, one could evaluate two surface integrals, one using the half-disk lying in the xz-plane, $\mathcal{S}_1$, and the other half-disk lying in the xy-plane, $\mathcal{S}_2$, since these share the same boundary as the portion of the sphere used above. Then

$$\int_{\partial\mathcal{S}} x\,\mathrm{d}x + (x-2yz)\,\mathrm{d}y + \left(x^2+z^4\right)\mathrm{d}z = \iint_{\mathcal{S}_1} \nabla\times\mathbf{F}\cdot\mathbf{n}_1\,\mathrm{d}S + \iint_{\mathcal{S}_2} \nabla\times\mathbf{F}\cdot\mathbf{n}_2\,\mathrm{d}S$$

$$\mathcal{S}_1 : g(x,y,z) = y \implies \nabla g = \mathbf{j} \text{ project onto } xz\text{-plane so that } \mathbf{p}=\mathbf{j} \implies |\nabla g\cdot\mathbf{p}| = 1$$

$$\nabla\times\mathbf{F}\cdot\frac{+\nabla g}{|\nabla g\cdot\mathbf{p}|} = \langle 2y, -2x, 1\rangle\cdot\langle 0,1,0\rangle = -2x$$

$$\iint_{\mathcal{S}_1} \nabla\times\mathbf{F}\cdot\mathbf{n}_1\,\mathrm{d}S = \int_{-1}^{1}\int_0^{\sqrt{1-x^2}} -2x\,\mathrm{d}z\,\mathrm{d}x = -2\int_{-1}^{1} x\sqrt{1-x^2}\,\mathrm{d}x = 0 \qquad \text{(odd integrand)}$$

$$\mathcal{S}_2 : g(x,y,z) = z \implies \nabla g = \mathbf{k} \text{ project onto } xy\text{-plane so that } \mathbf{p}=\mathbf{k} \implies |\nabla g\cdot\mathbf{p}| = 1$$

$$\nabla\times\mathbf{F}\cdot\frac{+\nabla g}{|\nabla g\cdot\mathbf{p}|} = \langle 2y, -2x, 1\rangle\cdot\langle 0,0,1\rangle = 1$$

$$\iint_{\mathcal{S}_2} \nabla\times\mathbf{F}\cdot\mathbf{n}_2\,\mathrm{d}S = \int_{-1}^{1}\int_0^{\sqrt{1-x^2}} 1\,\mathrm{d}y\,\mathrm{d}x = \text{area semi-circle of radius } 1 = \frac{\pi}{2}$$

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