1. [2350/041724 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

   (a) $\displaystyle\int_1^2 \int_0^{\sqrt{4-y^2}} 2x^2y^2\,\mathrm{d}x\,\mathrm{d}y = \int_0^{\sqrt{3}} \int_1^{\sqrt{4-x^2}} 2x^2y^2\,\mathrm{d}y\,\mathrm{d}x.$

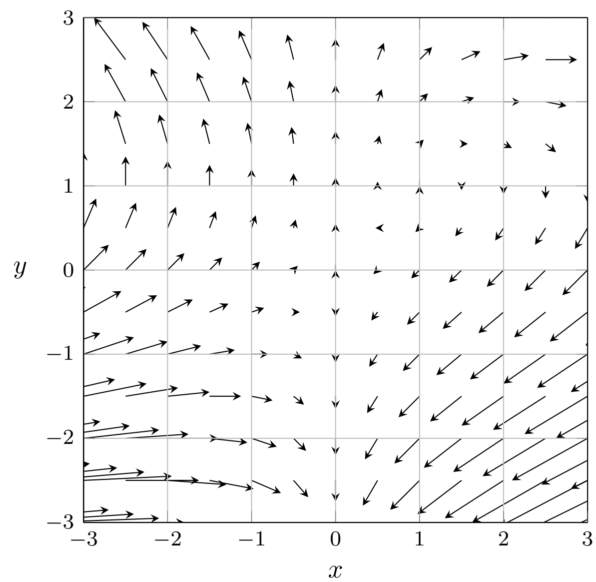
   (b) $\displaystyle\int_{-\sqrt{8-x^2}}^{x} \int_{-2}^{0} \sqrt{x^2+y^2}\,\mathrm{d}y\,\mathrm{d}x + \int_{-\sqrt{8-x^2}}^{-x} \int_{0}^{2} \sqrt{x^2+y^2}\,\mathrm{d}y\,\mathrm{d}x = \int_0^{2\sqrt{2}} \int_{5\pi/4}^{7\pi/4} r\,\mathrm{d}\theta\,\mathrm{d}r$

   (c) The surface area of the first octant portion of the hyperboloid of two sheets described by $x^2+y^2-z^2=-9,\ 3\le z\le 5$, is

$$\int_0^4 \int_0^{\sqrt{16-x^2}} \frac{\sqrt{2x^2+2y^2+9}}{\sqrt{x^2+y^2+9}}\,\mathrm{d}y\,\mathrm{d}x$$

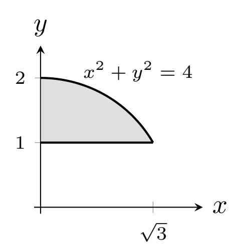
   (d) The points $(x,y,z)=(-1,1,-\sqrt{2})$ and $(\rho,\theta,\phi)=\left(2,-\frac{\pi}{4},\frac{3\pi}{4}\right)$ describe the same point in space.

   (e) The vector field $\mathbf{V}=(x-xy)\,\mathbf{i}+(x-y)\,\mathbf{j}$ is shown in the accompanying figure.



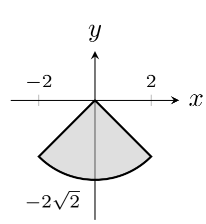
SOLUTION:

(a) **TRUE** Here is the region of integration.



(b) **FALSE** The bounds for rectangular coordinates integrals are incorrect (variables appear in the outer integrals) and the polar coordinate version does not include the Jacobian, r. The figure below gives the region and the correct integrals are

$$\int_{-2}^{0} \int_{-\sqrt{8-x^2}}^{x} \sqrt{x^2+y^2}\,\mathrm{d}y\,\mathrm{d}x + \int_0^2 \int_{-\sqrt{8-x^2}}^{-x} \sqrt{x^2+y^2}\,\mathrm{d}y\,\mathrm{d}x = \int_0^{2\sqrt{2}} \int_{5\pi/4}^{7\pi/4} r^2\,\mathrm{d}\theta\,\mathrm{d}r$$



(c) **TRUE** We project the surface onto the xy-plane so that $\mathbf{p} = \mathbf{k}$ and the region of integration is the first quadrant portion of the circle of radius 3.

$$g(x,y,z) = x^2 + y^2 - z^2$$
$$\nabla g = \langle 2x, 2y, -2z\rangle$$
$$\|\nabla g\| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2\sqrt{x^2 + y^2 + z^2}$$
$$|\nabla g \cdot \mathbf{p}| = |-2z| = 2z \qquad (z \geq 0)$$

Use the surface to eliminate z since integration occurs in the xy-plane. This gives the Jacobian as

$$\frac{\|\nabla g\|}{|\nabla g \cdot \mathbf{p}|} = \frac{2\sqrt{x^2+y^2+z^2}}{2z} = \frac{\sqrt{2x^2+2y^2+9}}{\sqrt{x^2+y^2+9}}$$

giving the surface area we seek as

$$\int_0^4 \int_0^{\sqrt{16-x^2}} \frac{\sqrt{2x^2+2y^2+9}}{\sqrt{x^2+y^2+9}}\, \mathrm{d}y\, \mathrm{d}x$$

(d) **FALSE** The actual spherical coordinates corresponding to $(x,y,z) = (-1, 1, -\sqrt{2})$ are $(\rho, \theta, \phi) = \left(2, \frac{3\pi}{4}, \frac{3\pi}{4}\right)$.

(e) **FALSE** The figure shows $-\mathbf{V}$.

■

2. [2350/041724 (16 pts)] A wire with charge density $q(x,y) = y^2$ is in the shape of the graph of $y = e^x, 0 \leq x \leq 1$. Find the total charge on wire.

SOLUTION:

The total charge on the wire is given by the line integral $\int_{\mathcal{C}} q(x,y)\, \mathrm{d}s$. Parameterize the curve (wire) as $\mathbf{r}(t) = \langle t, e^t\rangle, 0 \leq t \leq 1$ so that $\mathbf{r}'(t) = \langle 1, e^t\rangle$ and $\|\mathbf{r}'(t)\| = \sqrt{1+e^{2t}}$. Then

$$\text{Total charge} = \int_{\mathcal{C}} q(x,y)\, \mathrm{d}s = \int_0^1 e^{2t}\sqrt{1+e^{2t}}\, \mathrm{d}t \qquad \left(u = 1 + e^{2t}\right)$$
$$= \frac{1}{2}\int_2^{e^2+1} u^{1/2}\, \mathrm{d}u = \frac{1}{3}u^{3/2}\Big|_2^{e^2+1} = \frac{1}{3}\left[\left(e^2+1\right)^{3/2} - 2\sqrt{2}\right]$$

■

3. [2350/041724 (34 pts)] A thin metal plate is in the shape, $\mathcal{R}$, of a triangle with vertices $(0,0), (1,2), (2,3)$. The metal's density is given by $\delta(x,y) = \sqrt{4x-2y}(-2x+2y)$. We need to find the coordinates of the center of mass of the plate. We will use the Change of Variables theorem to accomplish this endeavor using the linear transformation $u = 4x - 2y, v = -2x + 2y$.

(a) (6 pts) Find the transformations of the vertices of the triangle.

(b) (6 pts) Recalling that linear transformations take lines into lines, draw the region of integration, $\mathcal{S}$, in the uv-plane, using the information from part (a). Be sure to label important points.

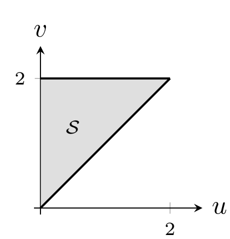
(c) (18 pts) Find the mass of the plate.

(d) (4 pts) Given that the moments with respect to the x- and y-axes are $\frac{104\sqrt{2}}{135}$ and $\frac{64\sqrt{2}}{135}$, respectively, find the coordinates of the center of mass of the plate. Note: $104 = (8)(13); 135 = (3)(45)$.

SOLUTION:

(a) $(u,v) = T(x,y) = (4x-2y, -2x+2y) \implies (0,0) \to (0,0);\ (1,2) \to (0,2);\ (2,3) \to (2,2)$

(b) Sketch.



(c) To use the change variables theorem, we need to find x and y in terms of u and v as well as the Jacobian of the transformation, $J(u,v)$. Adding $u = 4x - 2y$ to $v = -2x + 2y$ gives $x = \frac{1}{2}(u+v)$ while adding $u = 4x - 2y$ to $2v = -4x + 4y$ gives $y = \frac{1}{2}(u+2v)$. Then

$$J(u,v) = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{vmatrix} = \frac{1}{4}.$$

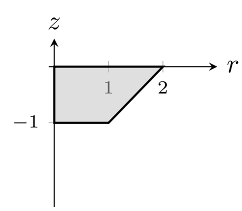
Thus,

$$\begin{aligned}
\text{Mass} = M &= \iint_{\mathcal{R}} \sqrt{4x-2y}(-2x+2y)\,\mathrm{d}A = \iint_{\mathcal{S}} \delta(u,v)|J(u,v)|\,\mathrm{d}u\,\mathrm{d}v \\
&= \int_0^2 \int_0^v \sqrt{u}\,(v)\left(\frac{1}{4}\right)\mathrm{d}u\,\mathrm{d}v \\
&= \frac{1}{4}\int_0^2 v\left(\frac{2}{3}u^{3/2}\right)\Bigg|_0^v \mathrm{d}v \\
&= \frac{1}{6}\int_0^2 v^{5/2}\,\mathrm{d}v \\
&= \frac{1}{6}\left(\frac{2}{7}\right)v^{7/2}\Big|_0^2 = \frac{1}{21}\left(2^3\right)\left(2^{1/2}\right) = \frac{8\sqrt{2}}{21}
\end{aligned}$$

(d)

$$x_{CM} = \frac{M_y}{M} = \frac{\frac{64\sqrt{2}}{135}}{\frac{8\sqrt{2}}{21}} = \frac{56}{45} \qquad y_{CM} = \frac{M_x}{M} = \frac{\frac{104\sqrt{2}}{135}}{\frac{8\sqrt{2}}{21}} = \frac{91}{45}$$

4. [2350/041724 (20 pts)] We need to evaluate $I = \iiint_{\mathcal{E}} z\left(x^2+y^2\right)^{3/2}\,\mathrm{d}V$ where the projection of $\mathcal{E}$ onto an rz-plane (constant θ) is shown in the following figure. The three dimensional solid region $\mathcal{E}$ is below the *second quadrant* of the xy-plane. In the parts that follow, set up, **do not evaluate**, integral(s) to compute I using the order of integration shown. Your limits must give the solid region as described, not using any potential symmetries. Simplify your integrands.



(a) $\mathrm{d}\theta\,\mathrm{d}r\,\mathrm{d}z$

(b) $\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$

(c) $\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$

<u>**SOLUTION:**</u>

(a)

$$I = \int_{-1}^{0}\int_{0}^{z+2}\int_{\pi/2}^{\pi} zr^4\,\mathrm{d}\theta\,\mathrm{d}r\,\mathrm{d}z$$

(b)

$$I = \int_{\pi/2}^{\pi}\int_{\pi/2}^{3\pi/4}\int_0^{2/(\sin\phi-\cos\phi)} \rho^6\sin^4\phi\cos\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta + \int_{\pi/2}^{\pi}\int_{3\pi/4}^{\pi}\int_0^{-\sec\phi} \rho^6\sin^4\phi\cos\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$$

(c)

$$I = \int_{\pi/2}^{\pi} \int_0^1 \int_{-1}^0 zr^4 \, \mathrm{d}z \, \mathrm{d}r \, \mathrm{d}\theta + \int_{\pi/2}^{\pi} \int_1^2 \int_{r-2}^0 zr^4 \, \mathrm{d}z \, \mathrm{d}r \, \mathrm{d}\theta$$

Alternatively,

$$I = \int_{\pi/2}^{\pi} \int_0^2 \int_{-1}^0 zr^4 \, \mathrm{d}z \, \mathrm{d}r \, \mathrm{d}\theta - \int_{\pi/2}^{\pi} \int_1^2 \int_{-1}^{r-2} zr^4 \, \mathrm{d}z \, \mathrm{d}r \, \mathrm{d}\theta$$

■

5. [2350/041724 (20 pts)] Let $\psi(x, y)$ be a function with continuous second partial derivatives and let $\mathbf{V} = \mathbf{k} \times \nabla\psi$ be a two-dimensional vector field.

   (a) (7 pts) Compute the divergence of $\mathbf{V}$.

   (b) (7 pts) Compute the curl of $\mathbf{V}$.

   (c) (3 pts) Is $\mathbf{V}$ incompressible? Justify your answer.

   (d) (3 pts) Is $\mathbf{V}$ irrotational? Justify your answer.

SOLUTION:

Note that

$$\mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ \partial\psi/\partial x & \partial\psi/\partial y & 0 \end{vmatrix} = -\mathbf{i}\,\frac{\partial\psi}{\partial y} + \mathbf{j}\,\frac{\partial\psi}{\partial x}$$

(a)

$$\nabla \cdot \mathbf{V} = \frac{\partial}{\partial x}\left(-\frac{\partial\psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial x}\right) = -\frac{\partial^2\psi}{\partial x \partial y} + \frac{\partial^2\psi}{\partial y \partial x} = -\frac{\partial^2\psi}{\partial x \partial y} + \frac{\partial^2\psi}{\partial x \partial y} = 0$$

(b)

$$\nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -\partial\psi/\partial y & \partial\psi/\partial x & 0 \end{vmatrix} = \left(\frac{\partial\, 0}{\partial y} - \frac{\partial^2\psi}{\partial z \partial x}\right)\mathbf{i} + \left(-\frac{\partial^2\psi}{\partial z \partial y} - \frac{\partial\, 0}{\partial x}\right)\mathbf{j} + \left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial\psi}{\partial y^2}\right)\mathbf{k} = \nabla^2\psi\,\mathbf{k}$$

(c) $\mathbf{V}$ is incompressible since $\nabla \cdot \mathbf{V} = 0$

(d) Since $\nabla^2\psi$ is not necessarily 0, $\nabla \times \mathbf{V}$ is not necessarily 0, implying that $\mathbf{V}$ is not irrotational.

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