1. [2350/031324 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

   (a) The domain of $w(x,y,z) = \ln(x+1)\sqrt{z^2-y}$ is $\left\{(x,y,z) \in \mathbb{R}^3 \,\middle|\, x > 0, y \le z^2\right\}$.

   (b) The smallest (most negative) rate of change of $f(x,y,z) = xyz$ at the point $(1,2,3)$ is -7.

   (c) The function $g(x,y) = \begin{cases} \frac{x^2-4y^2}{x+2y} & (x,y) \neq (2,-1) \\ -4 & (x,y) = (2,-1) \end{cases}$ is continuous at $(2,-1)$.

   (d) The tangent plane to the surface $yz + y - xe^z - e^{x+1} = 1$ at the point $(-1,1,0)$ is $-2x + y + 2z = 3$.

   (e) The level curves of any plane that is neither vertical nor horizontal are lines.

SOLUTION:

(a) **FALSE** The domain is $\left\{(x,y,z) \in \mathbb{R}^3 \,\middle|\, x > -1, y \le z^2\right\}$.

(b) **TRUE** $\nabla f(x,y,z) = \langle yz, xz, xy\rangle \implies \nabla f(1,2,3) = \langle 6,3,2\rangle$. The minimum rate of change of the function, that is, the most negative directional derivative, is $-\|-\nabla f(1,2,3)\| = -\sqrt{6^2+3^2+2^2} = -7$.

(c) **FALSE**

$$\lim_{(x,y)\to(2,-1)} \frac{x^2-4y^2}{x+2y} = \lim_{(x,y)\to(2,-1)} \frac{(x+2y)(x-2y)}{x+2y} = \lim_{(x,y)\to(2,-1)} x - 2y = 2 - (2)(-1) = 4 \neq -4 = g(2,-1)$$

This implies that $g(x,y)$ is not continuous at $(2,-1)$.

(d) **TRUE**

$$F(x,y,z) = yz + y - xe^z - e^{x+1} \implies \nabla F(x,y,z) = \left\langle -e^z - e^{x+1}, z+1, y - xe^z \right\rangle$$
$$\nabla F(-1,1,0) = \langle -e^0 - e^{-1+1}, 0+1, 1-(-1)e^0\rangle = \langle -2,1,2\rangle$$
$$-2[x-(-1)] + 1(y-1) + 2(z-0) = 0$$
$$-2x + y + 2z = 3$$

(e) **TRUE** The general equation of a plane is $ax + by + cz = d$. The plane is not vertical if $c \neq 0$ and the level curves are given by $ax + by = d - cz_0$, which are lines. The plane is not horizontal as long at least one of a or b is nonzero. For example, if $a = 0, b \neq 0$ we have $by = d - cz_0$, which is a line.

■

2. [2350/031324 (18 pts)] Parts (a) and (b) are not related.

   (a) (10 pts) Find the simplified quadratic (second degree) Taylor polynomial, T_2, for $f(x,y) = \sin(xy)$ centered at $(\pi,-1)$.

   (b) (8 pts) A group of your friends has a second degree Taylor polynomial for some function, $f(x,y)$, centered at the point $\left(\frac{1}{2},1\right)$. They want to use the polynomial to approximate the function in the region $\left|x-\frac{1}{2}\right| \le \frac{1}{2}, |y-1| \le 1$. In addition, they have provided you with the following information:

$$f_{xxx} = -3xy \qquad f_{xxy} = 3xy \qquad f_{xyy} = -7xy \qquad f_{yyy} = 4xy$$

   Tell your friends how much error they can expect in the approximation.

SOLUTION:

(a)

$$f_x(x,y) = y\cos(xy) \implies f_x(\pi,-1) = 1 \qquad f_y(x,y) = x\cos(xy) \implies f_y(\pi,-1) = -\pi$$
$$f_{xx}(x,y) = -y^2\sin(xy) \implies f_{xx}(\pi,-1) = 0 \qquad f_{yy} = -x^2\sin(xy) \implies f_{yy}(\pi,-1) = 0$$
$$f_{xy}(x,y) = \cos(xy) - xy\sin(xy) \implies f_{xy}(\pi,-1) = -1$$

Then

$$
\begin{aligned}
T_2(x,y) &= f(\pi,-1) + f_x(\pi,-1)(x-\pi) + f_y(\pi,-1)(y+1) \\
&\quad + \frac{1}{2!}\left[f_{xx}(\pi,-1)(x-\pi)^2 + 2f_{xy}(\pi,-1)(x-\pi)(y+1) + f_{yy}(\pi,-1)(y+1)^2\right] \\
&= 0 + 1(x-\pi) - \pi(y+1) + \frac{1}{2!}\left[0(x-\pi)^2 + 2(-1)(x-\pi)(y+1) + 0(y+1)^2\right] \\
&= (x-\pi) - \pi(y+1) - (x-\pi)(y+1) \\
&= x - \pi - \pi y - \pi - xy - x + \pi y + \pi \\
&= -xy - \pi
\end{aligned}
$$

(b) We need to bound the third derivatives on the region $\left|x - \frac{1}{2}\right| \le \frac{1}{2}, |y-1| \le 1$ which is equivalent to $0 \le x \le 1, 0 \le y \le 2$. On this rectangle, xy attains its largest value of 2 at $x = 1, y = 2$ since xy is an increasing function of both x and y. Using the given information on the third order derivatives, we have, on the given region,

$$
|f_{xxx}| \le 6 \qquad |f_{xxy}| \le 6 \qquad |f_{xyy}| \le 14 \qquad |f_{yyy}| \le 8
$$

$$
M = \max\{6, 6, 14, 8\} = 14
$$

$$
|E(x,y)| \le \frac{14}{3!}\left(\left|x - \tfrac{1}{2}\right| + |y-1|\right)^3 = \frac{14}{6}\left(\tfrac{1}{2} + 1\right)^3 = \frac{7}{3}\left(\frac{27}{8}\right) = \frac{63}{8}
$$

Your friends can expect the error to be no more than $\frac{63}{8}$.

Note that the following tighter bound can be found, which is also an acceptable answer. In what follows, x and y are confined to the given intervals and $xy \le 2$ on those intervals.

$$
\begin{aligned}
|E(x,y)| &\le \frac{1}{3!}\left[\left|f_{xxx}(x,y)\right|\left|x-\tfrac{1}{2}\right|^3 + 3\left|f_{xxy}(x,y)\right|\left|x-\tfrac{1}{2}\right|^2|y-1| + 3\left|f_{xyy}(x,y)\right|\left|x-\tfrac{1}{2}\right||y-1|^2 + \left|f_{yyy}(x,y)\right||y-1|^3\right] \\
&= \frac{1}{6}\left[|-3xy|\left(\tfrac{1}{2}\right)^3 + 3|3xy|\left(\tfrac{1}{2}\right)^2(1) + 3|-7xy|\left(\tfrac{1}{2}\right)(1)^2 + |4xy|(1)^3\right] \\
&= \frac{1}{6}\left[6\left(\tfrac{1}{8}\right) + 3(6)\left(\tfrac{1}{4}\right) + 3(14)\left(\tfrac{1}{2}\right) + 8\right] = \frac{1}{6}\left[\tfrac{3}{4} + \tfrac{18}{4} + \tfrac{84}{4} + \tfrac{32}{4}\right] = \tfrac{137}{24}
\end{aligned}
$$

■

3. [2350/031324 (34 pts)] Kalkk3 Regional Park consists of the boundary and interior of the triangle described by $x = 0, y = 0$ and $x + y = 9$. The elevation of the park above/below mean sea level ($h = 0$) is given by $h(x,y) = 2xy - \frac{2}{3}x^3 - y^2 + 100$. The editors of a travel brochure for the park want information about the various landforms in the park. Park surveyors have told you the following: maximum and minimum elevations along the southern ($y = 0$) border are 100 at $(0,0)$ and -386 at $(9,0)$, respectively, and along the western boundary ($x = 0$) they are 100 at $(0,0)$ and 19 at $(0,9)$.

(a) (15 pts) Are there any saddles or local hills or valleys in the park? If so, what are their elevations and locations?

(b) (15 pts) Use Lagrange Multipliers to find the maximum and minimum elevations and their locations along the remaining border, $x + y = 9$.

(c) (4 pts) If visitors to the park want to explore the highest and lowest points in the park, what points should they visit and what will their elevation be there?

SOLUTION:

(a) Find the critical points:

$$
h_x = 2y - 2x^2 \qquad h_y = 2x - 2y
$$

$$
h_{xx} = -4x \qquad h_{xy} = 2 \qquad h_{yy} = -2
$$

$$
2y - 2x^2 = 0 \tag{1}
$$

$$
2x - 2y = 0 \tag{2}
$$

Eq. (2) implies that $y = x$ and using this in Eq. (1) yields $2x(1-x) = 0 \implies x = 0, 1$, giving critical points of $(0,0)$ and $(1,1)$. Applying the Second Derivatives Test we have

$$D(x,y) = (-4x)(-2) - 2^2 = 8x - 4$$

$$D(0,0) = -4 < 0 \implies (0,0) \text{ is a saddle point with elevation } h(0,0) = 100$$

$$D(1,1) = 4 > 0, h_{yy}(1,1) < 0 \implies h(1,1) = \frac{301}{3} \text{ is a local maximum/hill}$$

(b) The objective function is $h(x,y) = 2xy - \frac{2}{3}x^3 - y^2 + 100$ and the constraint is $g(x,y) = x + y = 9$. We have $g_x = g_y = 1$ leading to

$$2y - 2x^2 = \lambda \tag{3}$$

$$2x - 2y = \lambda \tag{4}$$

Combining Eq. (3) and Eq. (4) gives $y - x^2 = x - y \implies y = \frac{1}{2}\left(x^2 + x\right)$. Using this in the constraint

$$x + \frac{1}{2}\left(x^2 + x\right) = 9$$

$$\frac{1}{2}x^2 + \frac{3}{2}x - 9 = 0$$

$$x^2 + 3x - 18 = 0$$

$$(x+6)(x-3) = 0 \implies x = -6, 3 \quad \text{(only 3 is relevant to the problem here)}$$

The critical point is thus $(3,6)$ giving an elevation of $h(3,6) = 82$. The values at the endpoints have been provided by the surveyors. Thus, the maximum elevation on this border is 82 at $(3,6)$ and the minimum is -386 at $(9,0)$.

(c) The highest elevation in the park is $\frac{301}{3}$ at $(1,1)$ and the lowest is -386 at $(9,0)$.

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4. [2350/031324 (22 pts)] The temperature in a region of space is given by $T(x,y,z) = 1000 + x^2 + y^2 + z^2$. The super-duper Rate-O-Change meter on board your spaceship gives a readout of the instantaneous rate of change of temperature with respect to any variable you enter into it. To receive credit, you must use Calculus 3 concepts to answer this question. Be careful with your notation.

(a) (10 pts) Suppose you know the following information about the path of your spaceship: $\mathbf{r}(1) = 2\,\mathbf{i} + \mathbf{k}$, $\mathbf{r}'(1) = 2\,\mathbf{i} + 2\pi\,\mathbf{j}$. What does your meter read when you enter t into it and you are at the point $(2,0,1)$?

(b) (12 pts) Now suppose your spaceship's position is given by $x(u,v,w) = v^2 + w^2$, $y(u,v,w) = \ln(uv)$, $z(u,v,w) = e^{2u+4v}$. What does your meter read if you enter u into it when $u = 2$, $v = \frac{1}{2}$ and $w = 1$?

SOLUTION:

(a) You are at the point $(2,0,1)$ when $t = 1$.

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{\partial T}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial T}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial T}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} = \nabla T[\mathbf{r}(t)] \cdot \mathbf{r}'(t)$$

$$\implies \left.\frac{\mathrm{d}T}{\mathrm{d}t}\right|_{t=1} = \nabla T\left[\mathbf{r}(1)\right] \cdot \mathbf{r}'(1) = \left.\langle 2x, 2y, 2z\rangle\right|_{(2,0,1)} \cdot \mathbf{r}'(1) = \langle 2(2), 2(0), 2(1)\rangle \cdot \langle 2, 2\pi, 0\rangle = 8$$

(b) When $u = 2, v = \frac{1}{2}$ and $w = 1$, $x = \frac{1}{4} + 1 = \frac{5}{4}$, $y = \ln\left[(2)\left(\frac{1}{2}\right)\right] = \ln 1 = 0$, $z = e^{2(2)+4(1/2)} = e^6$

$$\frac{\partial T}{\partial u} = \frac{\partial T}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial T}{\partial z}\frac{\partial z}{\partial u} = 2x(0) + 2y\left(\frac{v}{uv}\right) + 2z(2e^{2u+4v}) = \frac{2y}{u} + 4ze^{2u+4v}$$

$$\left.\frac{\partial T}{\partial u}\right|_{(u,v,w)=\left(2,\frac{1}{2},1\right)} = \frac{2(0)}{2} + 4e^6e^6 = 4e^{12}$$

■

5. [2350/031324 (16 pts)] The body mass index, B, as a function of weight, W (kg), and height, H (m), is given by $B = W/H^2$. Suppose that for a 2 m tall, 100 kg person you know that the height measurement is 0.01 m too high. If you want the body mass index to have an error no greater than 0.25, use differentials to determine the maximum error that can be present in the weight measurement.

SOLUTION:

The differential of B is

$$\mathrm{d}B = \frac{\partial B}{\partial W}\mathrm{d}W + \frac{\partial B}{\partial H}\mathrm{d}H = \frac{1}{H^2}\mathrm{d}W - \frac{2W}{H^3}\mathrm{d}H$$

Applying this at the point $(W, H) = (100, 2)$, noting that $\mathrm{d}H = 0.01$ and requiring $\mathrm{d}B$ to be bounded above by 0.25 yields

$$\mathrm{d}B = \frac{1}{4}\mathrm{d}W - \frac{(2)(100)}{2^3}\left(\frac{1}{100}\right) = \frac{1}{4}\mathrm{d}W - \frac{1}{4} \le \frac{1}{4} \implies \mathrm{d}W \le 2$$

The maximum error that can be present in the weight measurement is 2 kg. ■