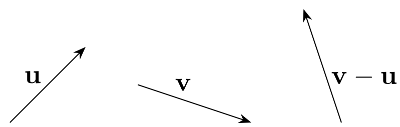
1. [2350/021424 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

   (a) If two nonzero vectors of equal magnitude, $\mathbf{A}$ and $\mathbf{B}$, are such that $\mathbf{A}\cdot\mathbf{B} = \|\mathbf{A}\times\mathbf{B}\|$, the the angle between the vectors is $\pi/4$.

   (b) The following figure is geometrically correct.



   (c) The vectors $\mathbf{u} = \mathbf{i} + 5\,\mathbf{j} - 2\,\mathbf{k}$, $\mathbf{v} = 3\,\mathbf{i} - \mathbf{j}$ and $\mathbf{w} = 5\,\mathbf{i} + 9\,\mathbf{j} - 4\,\mathbf{k}$ are coplanar.

   (d) The normal plane to the curve $\mathbf{r}(t) = t^3\,\mathbf{i} + 3t\,\mathbf{j} + t^4\,\mathbf{k}$ is parallel to the plane $3x + 3y - 4z = 2$ at the point $(-1, -3, 1)$.

   (e) $(\mathbf{a}\cdot\mathbf{b}) + \mathbf{c} = \mathbf{a}\cdot(\mathbf{b}+\mathbf{c})$

SOLUTION:

(a) **TRUE**

$$\mathbf{A}\cdot\mathbf{B} = \|\mathbf{A}\times\mathbf{B}\|$$
$$\|\mathbf{A}\|\|\mathbf{B}\|\cos\theta = \|\mathbf{A}\|\|\mathbf{B}\|\sin\theta$$
$$\cos\theta = \sin\theta$$
$$\tan\theta = 1 \implies \theta = \frac{\pi}{4}$$

(angle between vectors is defined to be between 0 and π)

(b) **FALSE** The third vector shown is $\mathbf{u} - \mathbf{v}$.

(c) **TRUE**

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} 1 & 5 & -2 \\ 3 & -1 & 0 \\ 5 & 9 & -4 \end{vmatrix} = 0$$

(d) **TRUE** $\mathbf{r}'(t) = \left\langle 3t^2, 3, 4t^3 \right\rangle$ is orthogonal to the normal plane of the curve. We want this to be parallel to the normal of the given plane, which is $\langle 3, 3, -4\rangle$, that is, we need to find a t such that $3t^2 = 3, 3 = 3, 4t^3 = -4 \implies t = -1$ giving the point $(-1, -3, 1)$.

(e) **FALSE** The expression on the left is meaningless (scalars and vectors cannot be added).

2. [2350/021424 (26 pts)] The following parts (a), (b) and (c) are not related.

   (a) (6 pts) Let $\mathbf{A}$ be the vector from the point $(2, -2, 4)$ to the point $(-2, 3, 5)$. Find a vector $\mathbf{D}$ having magnitude $\frac{1}{3}$ in the same direction as $\mathbf{A}$.

   (b) (12 pts) Let $\mathbf{a} = \mathbf{i} + 2\,\mathbf{j} - \mathbf{k}$ and $\mathbf{c} = \mathbf{j} - 2\,\mathbf{k}$.

      i. (6 pts) Find the area of the parallelogram formed by $\mathbf{a}$ and $\mathbf{c}$.

      ii. (6 pts) Find the vector projection of $\mathbf{a}$ onto $\mathbf{c}$.

   (c) (8 pts) Consider the equation $2x^2 + \alpha y^2 + z^2 - 2y - 4z + \beta = 0$.

      i. (2 pts) If $\alpha = 1$, find all values of β, if any, such that the equation describes an ellipsoid.

      ii. (6 pts) If $\alpha = -1$, find all values of β, if any, such that the equation describes a

      A. hyperboloid of one sheet B. hyperboloid of two sheets C. cone

SOLUTION:

(a) $\mathbf{A} = \langle -2-2, 3-(-2), 5-4\rangle = \langle -4, 5, 1\rangle$. Unit vector in direction of $\mathbf{A}$ is

$$\frac{\mathbf{A}}{\|\mathbf{A}\|} = \frac{\langle -4, 5, 1\rangle}{\sqrt{(-4)^2 + 5^2 + 1^2}} = \frac{1}{\sqrt{42}}\langle -4, 5, 1\rangle \implies \mathbf{D} = \frac{1}{3\sqrt{42}}\langle -4, 5, 1\rangle$$

(b) i. The area is $\|\mathbf{a}\times\mathbf{c}\|$.

$$\mathbf{a}\times\mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ 0 & 1 & -2 \end{vmatrix} = -3\,\mathbf{i} + 2\,\mathbf{j} + \mathbf{k} \implies \|\mathbf{a}\times\mathbf{c}\| = \sqrt{(-3)^2 + 2^2 + 1^2} = \sqrt{14}$$

ii.

$$\text{proj}_{\mathbf{c}}\mathbf{a} = \left(\frac{\mathbf{a}\cdot\mathbf{c}}{\mathbf{c}\cdot\mathbf{c}}\right)\mathbf{c} = \left(\frac{\langle 1,2,-1\rangle\cdot\langle 0,1,-2\rangle}{\langle 0,1,-2\rangle\cdot\langle 0,1,-2\rangle}\right)\langle 0,1,-2\rangle = \frac{4}{5}\langle 0,1,-2\rangle = \left\langle 0, \frac{4}{5}, -\frac{8}{5}\right\rangle$$

(c) i. If $\alpha = 1$, completing the square gives $2x^2 + (y-1)^2 + (z-2)^2 = 5 - \beta$. For this to describe an ellipsoid, $\beta < 5$.

ii. If $\alpha = -1$, completing the square gives $2x^2 - (y+1)^2 + (z-2)^2 = 3 - \beta$.

A. $3 - \beta > 0 \implies 3 > \beta$ B. $3 - \beta < 0 \implies 3 < \beta$ C. $3 - \beta = 0 \implies 3 = \beta$

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3. [2350/021424 (28 pts)] The path of a model rocket above ground (where $z = 0$) is given by

$$\mathbf{r}(t) = t^2\,\mathbf{i} + (\cos t + t\sin t)\,\mathbf{j} + (\sin t - t\cos t + 4\pi)\,\mathbf{k},\ t \geq 0$$

The label on the rocket engine claims that it has enough fuel to allow the rocket to travel $2\pi^2\sqrt{5}$ units along its path.

(a) (2 pts) Show that the velocity of the rocket is $2t\,\mathbf{i} + t\cos t\,\mathbf{j} + t\sin t\,\mathbf{k}$.

(b) (8 pts) When will the fuel run out, assuming that the fuel begins burning when $t = 0$?

(c) (4 pts) Find the coordinates of the rocket when the fuel has been completely consumed.

(d) (4 pts) Assuming the rocket is on the launchpad at $t = 0$, how far from the launchpad is the rocket when the fuel runs out?

(e) (10 pts) Once the fuel is consumed, the rocket initially moves in the direction it was moving when the fuel ran out but is acted on by a gravitational acceleration of $\mathbf{a} = \langle 0, 0, -10\rangle$. How long after the fuel is gone does it take the rocket to reach the ground? Hint: It may simplify things to consider a new time parameter, τ, equal to zero when the fuel is depleted.

SOLUTION:

(a) $\mathbf{v} = \mathbf{r}'(t) = 2t\,\mathbf{i} + (-\sin t + t\cos t + \sin t)\,\mathbf{j} + (\cos t + t\sin t - \cos t)\,\mathbf{k} = 2t\,\mathbf{i} + t\cos t\,\mathbf{j} + t\sin t\,\mathbf{k}$

(b) We need to know when the arc length of the rocket is the given value, that is, we need to find t such that $s(t) = \int_0^t \|\mathbf{r}'(u)\|\,\mathrm{d}u = 2\pi^2\sqrt{5}$. With $\|\mathbf{r}'(t)\| = \sqrt{4t^2 + t^2\cos^2 t + t^2\sin^2 t} = \sqrt{5t^2} = \sqrt{5}|t| = \sqrt{5}t$ since $t \geq 0$ we have

$$2\pi^2\sqrt{5} = \int_0^t \sqrt{5}u\,\mathrm{d}u = \frac{\sqrt{5}t^2}{2} \implies t = \pm 2\pi \quad (\text{choose positive value since } t \geq 0)$$

(c) The coordinates of the rocket when the fuel runs out are $\mathbf{r}(2\pi) = 4\pi^2\,\mathbf{i} + \mathbf{j} + 2\pi\,\mathbf{k}$.

(d) This is a direct application of the distance formula:

$$\text{distance} = \|\mathbf{r}(2\pi) - \mathbf{r}(0)\| = \left\|\left\langle 4\pi^2, 1, 2\pi\right\rangle - \langle 0, 1, 4\pi\rangle\right\| = \left\|\left\langle 4\pi^2, 0, -2\pi\right\rangle\right\| = \sqrt{16\pi^4 + 4\pi^2} = 2\pi\sqrt{4\pi^2 + 1}$$

(e) Let $\tau = 0$ be the time the fuel is depleted. We need to find the new path, $\mathbf{r}_{\text{new}}(\tau)$, where $\mathbf{r}_{\text{new}}(0) = \mathbf{r}(2\pi) = \left\langle 4\pi^2, 1, 2\pi\right\rangle$ and $\mathbf{v}_{\text{new}}(0) = \mathbf{r}'_{\text{new}}(0) = \mathbf{v}(2\pi) = \langle 4\pi, 2\pi, 0\rangle$.

$$\mathbf{v}_{\text{new}}(\tau) = \int \mathbf{v}'_{\text{new}}(\tau)\,\mathrm{d}\tau = \int \mathbf{a}(\tau)\,\mathrm{d}\tau = \int \langle 0, 0, -10\rangle\,\mathrm{d}\tau = \langle c_1, c_2, -10\tau + c_3\rangle$$

$$\mathbf{v}_{\text{new}}(0) = \langle c_1, c_2, c_3\rangle = \langle 4\pi, 2\pi, 0\rangle$$

$$\mathbf{v}_{\text{new}}(\tau) = \langle 4\pi, 2\pi, -10\tau\rangle$$

$$\mathbf{r}_{\text{new}}(\tau) = \int \mathbf{r}'_{\text{new}}(\tau)\,\mathrm{d}\tau = \int \mathbf{v}_{\text{new}}(\tau)\,\mathrm{d}\tau = \int \langle 4\pi, 2\pi, -10\tau\rangle\,\mathrm{d}\tau = \left\langle 4\pi\tau + k_1, 2\pi\tau + k_2, -5\tau^2 + k_3\right\rangle$$

$$\mathbf{r}_{\text{new}}(0) = \langle k_1, k_2, k_3\rangle = \left\langle 4\pi^2, 1, 2\pi\right\rangle$$

$$\mathbf{r}_{\text{new}}(\tau) = \left\langle 4\pi\tau + 4\pi^2, 2\pi\tau + 1, -5\tau^2 + 2\pi\right\rangle$$

The rocket will hit the ground when the z-component of the path equals 0. Thus

$$-5\tau^2 + 2\pi = 0 \implies \tau = \sqrt{\frac{2\pi}{5}}$$

4. [2350/021424 (20 pts)] Consider the two points $P(1,0,2)$ and $Q(-1,2,0)$ and the plane M described by $2x-4y+z=10$.

(a) (10 pts) Find the equation of the plane containing the points P and Q and perpendicular to the plane M. Write your answer in the standard form $ax+by+cz=d$.

(b) (10 pts) Find the symmetric equations of the line orthogonal to M and passing through point Q.

SOLUTION:

(a) Let N be the plane we seek. The vector $\overrightarrow{PQ}=\langle -1-1, 2-0, 0-2\rangle = \langle -2,2,-2\rangle$ lies in N. The normal, $\mathbf{n}=\langle 2,-4,1\rangle$, to the given plane M is parallel to N. A vector orthogonal to both $\overrightarrow{PQ}$ and $\mathbf{n}$ is $\overrightarrow{PQ}\times\mathbf{n}$ and will serve as the normal to N.

$$\overrightarrow{PQ}\times\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 2 & -2 \\ 2 & -4 & 1 \end{vmatrix} = -6\,\mathbf{i} - 2\,\mathbf{j} + 4\,\mathbf{k}$$

The equation of N is thus

$$-6(x-1)-2(y-0)+4(z-2)=0 \implies -6x-2y+4z=2 \implies 3x+y-2z=-1$$

(b) The direction of the sought after line is the normal to $M, 2\,\mathbf{i}-4\,\mathbf{j}+\mathbf{k}$, and a point on the line is $(-1,2,0)$. The parametric equations are $x=-1+2t, y=2-4t, z=t$ leading to the symmetric equations

$$\frac{x+1}{2}=\frac{y-2}{-4}=z$$

5. [2350/021424 (16 pts)] A particle is moving along the path $\mathbf{r}(t)=e^t\,\mathbf{i}+2\,\mathbf{j}+e^{-t}\,\mathbf{k}$.

(a) (4 pts) Without doing any computations, briefly explain why the unit binormal, $\mathbf{B}$, is always parallel to the y-axis.

(b) (12 pts) Recalling that the particle's velocity is a vector and its change, the acceleration, can be decomposed into two orthogonal components, answer the following questions.

i. (5 pts) Find the rate of change of the direction of the particle as a function of t.

ii. (7 pts) What is the speed of the particle at the point where its speed is not changing?

SOLUTION:

(a) The path lies in the plane $y=2$, thus $\mathbf{T}$ and $\mathbf{N}$ have only $\mathbf{i}$- and $\mathbf{k}$-components. Since $\mathbf{B}=\mathbf{T}\times\mathbf{N}$ is orthogonal to both $\mathbf{T}$ and $\mathbf{N}$, it is equal to $\pm\mathbf{j}$, which is parallel to the y-axis.

(b) The normal component of the acceleration gives the direction changes and the tangential component gives speed changes.

i. We need to find the normal component of the acceleration.

$$\mathbf{v}(t)=\mathbf{r}'(t)=e^t\mathbf{i}-e^{-t}\mathbf{k}$$

$$\mathbf{a}(t)=\mathbf{r}''(t)=e^t\,\mathbf{i}+e^{-t}\mathbf{k}$$

$$\mathbf{r}'(t)\times\mathbf{r}''(t)=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ e^t & 0 & -e^{-t} \\ e^t & 0 & e^{-t} \end{vmatrix} = -2\mathbf{j}$$

$$\|\mathbf{r}'(t)\times\mathbf{r}''(t)\| = \|-2\mathbf{j}\| = 2$$

$$\|\mathbf{r}'(t)\| = \sqrt{e^{2t}+e^{-2t}}$$

$$a_N(t)=\frac{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|}=\frac{2}{\sqrt{e^{2t}+e^{-2t}}}$$

ii. We need the tangential acceleration to solve this.

$$\mathbf{r}'(t)\cdot\mathbf{r}''(t)=\left(e^t\mathbf{i}-e^{-t}\mathbf{k}\right)\cdot\left(e^t\,\mathbf{i}+e^{-t}\mathbf{k}\right)=e^{2t}-e^{-2t}$$

$$a_T(t)=\frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{\|\mathbf{r}'(t)\|}=\frac{e^{2t}-e^{-2t}}{\sqrt{e^{2t}+e^{-2t}}}$$

The particle's speed is not changing when $a_T(t) = 0$.

$$\frac{e^{2t} - e^{-2t}}{\sqrt{e^{2t} + e^{-2t}}} = 0 \implies e^{2t} - e^{-2t} = 0 \implies e^{4t} = 1 \implies t = 0$$

The particle's speed at $t = 0$ is $\|\mathbf{v}(0)\| = \|\mathbf{i} - \mathbf{k}\| = \sqrt{2}$.

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