1. [2350/050823 (46 pts)] A wire is in the shape of the curve $\mathcal{C}$ given by $\mathbf{r}(t) = t\,\mathbf{i} + t\,\mathbf{j} + t^2\,\mathbf{k}$, $-1 \le t \le 2$.

   (a) [9 pts] Does the wire intersect the plane that contains the points $(1,2,0), (0,0,3), (0,-1,4)$? If so, find the point of intersection. If not, explain why not.

   (b) [5 pts] What is the curvature of the wire when $t = 0$?

   (c) [15 pts] If the charge density (Coulomb per unit length) on the wire is given by $q(x,y,z) = \dfrac{5xyz}{\sqrt{2+4xy}}$, find the total charge on the wire.

   (d) [17 pts] Answer the following questions if the wire is immersed in an electric field, $\mathbf{E}$, given by

   $$\mathbf{E} = -x\cos y\,\mathbf{i} + y\cos x\,\mathbf{j} + xye^{x-y+z^2}\,\mathbf{k}$$

   i. [2 pts] Is the electric field incompressible? Justify your answer.

   ii. [15 pts] Find the work done moving a charged particle along the wire. Hint: $\mathbf{E}$ is not irrotational.

SOLUTION:

(a) Two vectors that lie in the plane are $\mathbf{u} = \langle 1,2,0\rangle - \langle 0,0,3\rangle = \langle 1,2,-3\rangle$ and $\mathbf{v} = \langle 0,0,3\rangle - \langle 0,-1,4\rangle = \langle 0,1,-1\rangle$. Their cross product gives the normal, $\mathbf{n}$, to the plane as

$$\mathbf{n} = \mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -3 \\ 0 & 1 & -1 \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

The equation of the plane is $1(x-0) + 1(y-0) + 1(z-3) = 0 \implies x + y + z = 3$. To find the where the wire intersects the plane, we use the coordinates of the wire in the equation of the plane:

$$t + t + t^2 = 3 \implies t^2 + 2t - 3 = (t+3)(t-1) = 0 \implies t = -3, 1$$

Since $-1 \le t \le 2$, only $t = 1$ lies on the wire. The wire intersects the plane at $(1,1,1)$.

(b)

$$\mathbf{r}'(t) = \mathbf{i} + \mathbf{j} + 2t\,\mathbf{k} \implies \|\mathbf{r}'(t)\| = \sqrt{2+4t^2}$$

$$\mathbf{r}''(t) = 2\,\mathbf{k}$$

$$\mathbf{r}'\times\mathbf{r}'' = (\mathbf{i} + \mathbf{j} + 2t\,\mathbf{k}) \times 2\,\mathbf{k} = 2\,\mathbf{i} - 2\,\mathbf{j}$$

$$\kappa(t) = \frac{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\|2\,\mathbf{i} - 2\,\mathbf{j}\|}{\left(\sqrt{2+4t^2}\right)^3} = \frac{\sqrt{8}}{(2+4t^2)^{3/2}} \implies \kappa(0) = \frac{\sqrt{8}}{2^{3/2}} = 1$$

(c) We need to evaluate a scalar line integral.

$$q(\mathbf{r}(t)) = \frac{5t^4}{\sqrt{2+4t^2}}$$

$$\text{Total charge} = \int_{\mathcal{C}} q(x,y,z)\,\mathrm{d}s = \int_{-1}^{2} q(\mathbf{r}(t))\|\mathrm{r}'(t)\|\,\mathrm{d}t = \int_{-1}^{2} \frac{5t^4}{\sqrt{2+4t^2}}\sqrt{2+4t^2}\,\mathrm{d}t = t^5\Big|_{-1}^{2} = 2^5 - (-1)^5 = 33 \text{ Coulomb}$$

(d) i. No.

$$\nabla\cdot\mathbf{E} = \frac{\partial}{\partial x}(-x\cos y) + \frac{\partial}{\partial y}(y\cos x) + \frac{\partial}{\partial z}\left(xye^{x-y+z^2}\right) = -\cos y + \cos x + 2xyze^{x-y+z^2} \neq 0$$

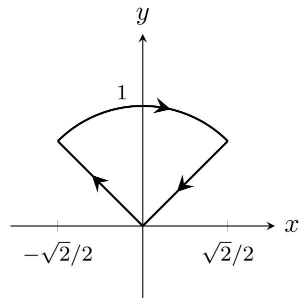
ii. We need to evaluate a vector line integral.

$$\mathbf{E}(\mathbf{r}(t)) = -t\cos t\,\mathbf{i} + t\cos t\,\mathbf{j} + t^2e^{t^4}\,\mathbf{k}$$

$$\text{Work} = \int_{\mathcal{C}} \mathbf{E}\cdot\mathrm{d}\mathbf{r} = \int_{-1}^{2} \mathbf{E}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,\mathrm{d}t = \int_{-1}^{2} \left(-t\cos t\,\mathbf{i} + t\cos t\,\mathbf{j} + t^2e^{t^4}\,\mathbf{k}\right)\cdot(\mathbf{i} + \mathbf{j} + 2t\,\mathbf{k})\,\mathrm{d}t$$

$$= \int_{-1}^{2} 2t^3e^{t^4}\,\mathrm{d}t \overset{u=t^4}{=} \frac{1}{2}\int_{1}^{16} e^u\,\mathrm{d}u = \frac{e}{2}\left(e^{15} - 1\right)$$

2. [2350/050823 (20 pts)] Consider the oriented curve, $\mathcal{C}$, shown in the figure (the curved portion is an arc of the unit circle). Compute the circulation of $\mathbf{V}$ on $\mathcal{C}$ where $\mathbf{V} = \left(-16y + \sin x^2\right)\mathbf{i} + \left(4e^y + 3x^2\right)\mathbf{j}$.



SOLUTION:

We don't want to compute the circulation directly (actually, we can't) so we will employ Green's theorem. We let $\mathcal{D}$ be the region inside the curve $\mathcal{C}$. To use Green's Theorem the curve must be oriented the opposite way from that given.

$$\begin{aligned}\text{Circulation} &= \oint_{\mathcal{C}} \mathbf{V}\cdot \mathrm{d}\mathbf{r} = -\oint_{-\mathcal{C}} \mathbf{V}\cdot \mathrm{d}\mathbf{r} = -\iint_{\mathcal{D}} \left[\frac{\partial}{\partial x}\left(4e^y + 3x^2\right) - \frac{\partial}{\partial y}\left(-16y + \sin x^2\right)\right] \mathrm{d}A \\
&= -\iint_{\mathcal{D}} (6x + 16)\,\mathrm{d}A \overset{\text{polar}}{=} -\int_{\pi/4}^{3\pi/4}\int_0^1 (6r\cos\theta + 16)\, r\,\mathrm{d}r\,\mathrm{d}\theta \\
&= -\int_{\pi/4}^{3\pi/4} \left(2r^3\cos\theta + 8r^2\right)\Big|_0^1 \mathrm{d}\theta = -\int_{\pi/4}^{3\pi/4} (2\cos\theta + 8)\,\mathrm{d}\theta \\
&= -\left(2\sin\theta + 8\theta\right)\Big|_{\pi/4}^{3\pi/4} = -\left[2\left(\frac{\sqrt{2}}{2}\right) + 6\pi - 2\left(\frac{\sqrt{2}}{2}\right) - 2\pi\right] = -4\pi\end{aligned}$$

3. [2350/050823 (22 pts)] Consider the three dimensional solid, $\mathcal{E}$, below the surface $\mathcal{S}_1 : \rho = 2\cos\phi$ and above the surface $\mathcal{S}_2 : \phi = \dfrac{\pi}{4}$. Let $\mathbf{F} = y\,\mathbf{i} + x\,\mathbf{j} + \left(6z\tan^{-1}\dfrac{y}{x}\right)\mathbf{k}$. Recall that $\tan\theta = \dfrac{y}{x}$.

(a) [2 pts] Name the surfaces, $\mathcal{S}_1$ and $\mathcal{S}_2$.

(b) [20 pts] Find the outward flux of $\mathbf{F}$ through the boundary of $\mathcal{E}$ ($\partial\mathcal{E} = \mathcal{S}_1 \cup \mathcal{S}_2$).

SOLUTION:

(a) $\mathcal{S}_1$ is a sphere and $\mathcal{S}_2$ is a cone.

(b) We use Gauss' Divergence theorem and evaluate the triple integral using spherical coordinates.

$$\nabla\cdot\mathbf{F} = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}\left(6z\tan^{-1}\frac{y}{x}\right) = 6\tan^{-1}\frac{y}{x} = 6\theta \text{ when converting to spherical coordinates}$$

$$\begin{aligned}\iint_{\partial\mathcal{E}} \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S &= \iiint_{\mathcal{E}} \nabla\cdot\mathbf{F}\,\mathrm{d}V = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{2\cos\phi} 6\theta\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta \\
&= 6\left(\int_0^{2\pi}\theta\,\mathrm{d}\theta\right)\int_0^{\pi/4} \frac{1}{3}\rho^3\Big|_0^{2\cos\phi} \sin\phi\,\mathrm{d}\phi \\
&= \theta^2\Big|_0^{2\pi}\left(8\int_0^{\pi/4}\cos^3\phi\sin\phi\,\mathrm{d}\phi\right) \qquad u = \cos\phi \\
&= 32\pi^2\int_{\sqrt{2}/2}^1 u^3\,\mathrm{d}u = 8\pi^2 u^4\Big|_{\sqrt{2}/2}^1 \\
&= 8\pi^2\left(1 - \frac{1}{4}\right) = 6\pi^2\end{aligned}$$

■

4. [2350/050823 (20 pts)] Compute $\int_{\mathcal{C}} P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z$ where $\mathbf{F} = \langle P, Q, R\rangle = y\,\mathbf{i} + x^2\,\mathbf{j} + z\,\mathbf{k}$ by evaluating an appropriate surface integral. $\mathcal{C}$ is the boundary of the portion of the plane $x + y + 5z = 1$ in the first octant, oriented counterclockwise when viewed from above.

SOLUTION:

We use Stokes' Theorem.

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & x^2 & z \end{vmatrix} = (2x-1)\,\mathbf{k}$$

The surface is $g(x, y, z) = x + y + 5z$ which we project onto the xy-plane giving $\mathcal{R}$ as the triangle with vertices $(0, 0), (1, 0), (0, 1)$ and $\mathbf{p} = \mathbf{k}$. The normal vector to the surface is $\nabla g = \mathbf{i} + \mathbf{j} + 5\,\mathbf{k}$ which is the proper orientation for the surface's normal vector given the orientation of the surface's boundary, $\mathcal{C}$. Furthermore, $|\nabla g \cdot \mathbf{p}| = 5$. Thus

$$\int_{\mathcal{C}} y\,\mathrm{d}x + x^2\,\mathrm{d}y + z\,\mathrm{d}z = \iint_{\mathcal{S}} \nabla \times \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}S = \iint_{\mathcal{R}} \nabla \times \mathbf{F} \cdot \frac{\nabla g}{|\nabla g \cdot \mathbf{p}|} = \int_0^1 \int_0^{1-x} \frac{\langle 0, 0, 2x-1\rangle \cdot \langle 1, 1, 5\rangle}{5} \mathrm{d}y\,\mathrm{d}x$$
$$= \int_0^1 \int_0^{1-x} (2x-1)\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 (2x-1)\,y\Big|_0^{1-x}\,\mathrm{d}x$$
$$= \int_0^1 \left(-2x^2 + 3x - 1\right)\mathrm{d}x = \left(-\frac{2}{3}x^2 + \frac{3}{2}x - x\right)\Bigg|_0^1 = -\frac{1}{6}$$

■

5. [2350/050823 (15 pts)] The temperature in a certain region of the xy plane is given by $T(x, y) = 2x^2 + 2y^2 + x - y$. You are walking along the circle $x^2 + y^2 = 8$. Use Lagrange Multipliers to find the maximum and minimum temperatures you experience during your walk.

SOLUTION:

We need to optimize $T(x, y) = 2x^2 + 2y^2 + x - y$ subject to the constraint $g(x, y) = x^2 + y^2 = 8$

$$T_x = 4x + 1 \qquad g_x = 2x$$
$$T_y = 4y - 1 \qquad g_y = 2y$$

We need to solve the simultaneous equations

$$4x + 1 = \lambda(2x) \tag{1}$$
$$4y - 1 = \lambda(2y) \tag{2}$$
$$x^2 + y^2 = 8 \tag{3}$$

Neither x nor y can be zero in (1) and (2), respectively, so we can divide (1) by x and (2) by y and equate the results.

$$\frac{4x+1}{2x} = \lambda = \frac{4y-1}{2y} \implies \cancel{4} + \frac{1}{x} = \cancel{4} - \frac{1}{y} \implies y = -x$$
$$x^2 + y^2 = x^2 + (-x)^2 = 2x^2 = 8 \implies x = \pm 2 \implies y = \mp 2$$
critical points are $(2, -2), (-2, 2)$
$$T(2, -2) = 2(2)^2 + 2(-2)^2 + 2 - (-2) = 20 \quad \text{and} \quad T(-2, 2) = 2(-2)^2 + 2(2)^2 - 2 - 2 = 12$$

The maximum temperature experienced is 20 and the minimum is 12. ■

6. [2350/050823 (27 pts)] On a separate page in your bluebook, write the letters (a) through (i) in a column. Then for the following questions, write the word **TRUE** or **FALSE** next to each letter, as appropriate. No partial credit given and no work need be shown. If you do any work to come up with your answers, please do it elsewhere - do not include it in your list of answers (this helps with grading).

(a) The flow along any curve $\mathcal{C}$ between the origin and $(r, \theta, z) = \left(\sqrt{2}, \frac{\pi}{4}, 1\right)$ of the vector field $\mathbf{F} = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$ is 1.

(b) The function $z = (x-1)(y+1)$ has a local maximum at $(1,-1)$.

(c) The second order Taylor polynomial centered at $(0,0)$ of the function $f(x,y) = xy$ is $f(x,y)$.

(d) The Extreme Value Theorem guarantees that $f(x,y) = \dfrac{xy}{1+x^2+y^2}$ attains an absolute maximum on the region $x^2+y^2<2$.

(e) The binormal vector, $\mathbf{B}$, of any curve lying entirely in the xz-plane is parallel to the y-axis.

(f) A particle traveling along the path $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + e^t\,\mathbf{k},\ t \geq 0$ never slows down.

(g) The function $f(x,y) = x^2 - xy + y^2$ increases the fastest at the point $(2,1)$ in the direction of $-3\,\mathbf{i}$.

(h) The level surfaces of $w(x,y,z) = x^2 - y^2 + 2z^2$ are hyperbolic paraboloids.

(i) The function $f(x,y) = \begin{cases} 0 & (x,y) = (0,0) \\ \dfrac{x^3y}{x^4+y^4} & (x,y) \neq (0,0) \end{cases}$ is continuous throughout $\mathbb{R}^2$.

SOLUTION:

(a) **TRUE** Since $\nabla \times \mathbf{F} = \mathbf{0}$ on a simply connected region ($\mathbb{R}^3$), $\mathbf{F}$ is conservative and a potential function, f, exists such that $\mathbf{F} = \nabla f$. By inspection, $f(x,y,z) = xyz$. The Fundamental Theorem for Line Integrals applies. Thus

$$\text{Flow} = \int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = f(1,1,1) - f(0,0,0) = 1$$

(b) **FALSE**

$$z_x = y + 1 = 0 \implies y = -1 \qquad z_y = x - 1 = 0 \implies x = 1 \implies (1,-1) \text{ is the only critical point}$$

$$z_{xx} = 0 \quad z_{yy} = 0 \quad z_{xy} = 1 \implies D(1,-1) = z_{xx}(1,-1)z_{yy}(1,-1) - \left[z_{xy}(1,-1)\right]^2 = -1 < 0 \implies (1,-1) \text{ is a saddle}$$

(c) **TRUE** $f_x = y, f_y = x, f_{xx} = 0, f_{yy} = 0, f_{xy} = 1$. Then

$$\begin{aligned} T_2(x,y) &= f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) + \frac{1}{2!}\left[f_{xx}(0,0)(x-0)^2 + 2f_{xy}(0,0)(x-0)(y-0) + f_{yy}(y-0)^2\right] \\ &= 0 + 0x + 0y + \frac{1}{2!}\left[0x^2 + 2(1)xy + 0y^2\right] \\ &= xy = f(x,y) \end{aligned}$$

(d) **FALSE** The function is continuous on the region (it is a rational function) but the region is not closed (it is bounded, however). Thus the Extreme Value Theorem guarantees nothing.

(e) **TRUE** The binormal vector is perpendicular to the osculating plane which in this case lies in the xz-plane. Vectors normal to this plane are parallel to the y-axis. Indeed $\mathbf{B} = \pm\mathbf{j}$.

(f) **TRUE** The magnitude of the tangential acceleration is always positive.

$$\mathbf{r}'(t) = \mathbf{i} + 2t\,\mathbf{j} + e^t\,\mathbf{k} \implies \|\mathbf{r}'(t)\| = \|\mathbf{v}(t)\| = \sqrt{1+4t^2+e^{2t}}$$

$$\mathbf{r}''(t) = 2\,\mathbf{j} + e^t\,\mathbf{k}$$

$$a_T = \frac{\mathrm{d}\|\mathrm{v}\|}{\mathrm{d}t} = \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{\|\mathbf{r}'(t)\|} = \frac{4+e^{2t}}{2\sqrt{1+4t^2+e^{2t}}} > 0 \text{ for all } t$$

(g) **FALSE**

$$\nabla f(x,y) = (2x-y)\,\mathbf{i} + (2y-x)\,\mathbf{j} \implies \nabla f(2,1) = 3\,\mathbf{i}$$

The function decreases the fastest in the direction of $-3\,\mathbf{i}$.

(h) **FALSE** The level surfaces are cones and hyperboloids of one and two sheets with axes along the y-axis $\left(x^2 - y^2 + 2z^2 = c\right)$.

(i) **FALSE** The function is defined throughout $\mathbb{R}^2$. Away from the origin it is continuous since it is a rational function. At the origin, however, the limit does not exist. To see this, approach the origin along the line $y = mx$. Then

$$\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^4+y^4} = \lim_{(x,mx)\to(0,0)} \frac{x^3(mx)}{x^4+(mx)^4} = \lim_{(x,mx)\to(0,0)} \frac{m}{1+m^4} = \frac{m}{1+m^4}$$

which depends on m, proving that the limit does not exist at the origin and therefore the function is not continuous at the origin and thus not continuous throughout $\mathbb{R}^2$.

■