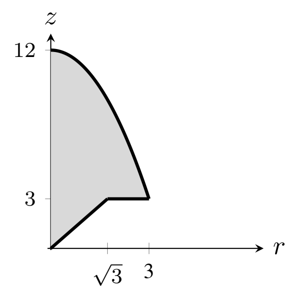
1. [2350/041923 (22 pts)] You are in charge of making three-dimensional stoppers for glass bottles. The density of the material from which the stoppers are made is $\delta(x, y, z) = x^2 + y^2 + 7$. A cross section of half of the stopper is depicted in the following rz–plane (constant θ plane). The curved portion of the stopper is $x^2 + y^2 + z = 12$.



(a) [8 pts] Set up, **but do not evaluate**, the appropriate integral(s) that will give the mass of the stopper using cylindrical coordinates and the order $\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$.

(b) [14 pts] Now suppose that only the curved portion of the stopper is replaced with a portion of the sphere of radius 3, centered at $(x, y, z) = (0, 0, 3)$, while the rest of the stopper's shape remains the same.

i. [6 pts] Make a sketch of this new stopper in the rz–plane (constant θ plane). Be sure to label important points in your sketch.

ii. [8 pts] Using spherical coordinates and the order $\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$, set up, **but do not evaluate**, the appropriate integral(s) to find the mass of this new stopper.

SOLUTION:

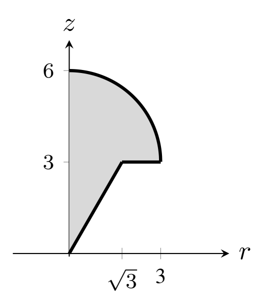
(a)

$$\text{Mass} = \int_0^{2\pi}\int_0^{\sqrt{3}}\int_{\sqrt{3}r}^{12-r^2} \left(r^2+7\right) r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta + \int_0^{2\pi}\int_{\sqrt{3}}^{3}\int_{3}^{12-r^2} \left(r^2+7\right) r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$$

Alternatively, integrating over the triangular portion and then the paraboloidal portion gives

$$\text{Mass} = \int_0^{2\pi}\int_0^{\sqrt{3}}\int_{\sqrt{3}r}^{3} \left(r^2+7\right) r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta + \int_0^{2\pi}\int_0^{3}\int_{3}^{12-r^2} \left(r^2+7\right) r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$$

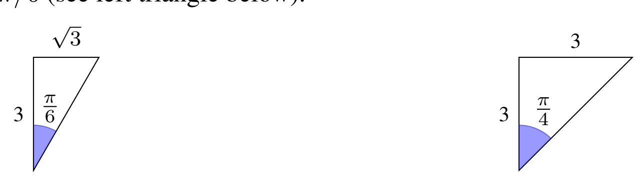
(b) i. Sketch of new stopper.



ii. The spherical coordinates equation for the given sphere is

$$\begin{aligned} x^2 + y^2 + (z-3)^2 &= 9 \\ x^2 + y^2 + z^2 - 6z + 9 &= 9 \\ x^2 + y^2 + z^2 &= 6z \\ \rho^2 &= 6\rho\cos\phi \\ \rho &= 6\cos\phi \end{aligned}$$

The horizontal line is $z = 3 \implies \rho\cos\phi = 3 \implies \rho = 3\sec\phi$ in spherical coordinates while the sloped line's equation is spherical coordinates is $\phi = \pi/6$ (see left triangle below).



With aid of both triangles, we have

$$\text{Mass} = \int_0^{2\pi}\int_0^{\pi/6}\int_0^{6\cos\phi} \left(\rho^2\sin^2\phi + 7\right)\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta + \int_0^{2\pi}\int_{\pi/6}^{\pi/4}\int_{3\sec\phi}^{6\cos\phi} \left(\rho^2\sin^2\phi + 7\right)\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$$

Alternatively, integrating over the triangular portion and then the spherical portion gives

$$\text{Mass} = \int_0^{2\pi}\int_0^{\pi/6}\int_0^{3\sec\phi} \left(\rho^2\sin^2\phi + 7\right)\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta + \int_0^{2\pi}\int_0^{\pi/4}\int_{3\sec\phi}^{6\cos\phi} \left(\rho^2\sin^2\phi + 7\right)\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$$

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2. [2350/041923 (20 pts)] The charge density on a metal plate is given by $q(x,y) = \sqrt{xy}$ coulombs per square meter. The plate occupies the first quadrant region bounded by $xy = 1$, $xy = 9$, $y/x = 1$, and $y/x = 4$. Use the change of variables $u = xy$ and $v = y/x$ to find the total charge on the plate. Be sure to include appropriate units.

SOLUTION:

$$u = xy, v = y/x \implies uv = y^2 \implies y = \sqrt{uv} = u^{1/2}v^{1/2} \quad \text{and} \quad u/v = x^2 \implies x = \sqrt{u/v} = u^{1/2}v^{-1/2}$$

(positive square roots since the plate is in the first quadrant)

$$J(u,v) = \begin{vmatrix} \frac{1}{2}u^{-1/2}v^{-1/2} & -\frac{1}{2}u^{1/2}v^{-3/2} \\ \frac{1}{2}u^{-1/2}v^{1/2} & \frac{1}{2}u^{1/2}v^{-1/2} \end{vmatrix} = \frac{1}{2v}$$

The region of integration is the rectangle $1 \le u \le 9$, $1 \le v \le 4$. Thus

$$\begin{aligned} \text{Charge} &= \int_1^9\int_1^4 \sqrt{u}\left|\frac{1}{2v}\right|\,\mathrm{d}v\,\mathrm{d}u \\ &= \frac{1}{2}\left(\int_1^9 u^{1/2}\,\mathrm{d}u\right)\left(\int_1^4 \frac{1}{v}\,\mathrm{d}v\right) \\ &= \frac{1}{2}\left(\frac{2}{3}u^{3/2}\Big|_1^9\right)\left(\ln|v|\Big|_1^4\right) \\ &= \frac{1}{3}\left(9^{3/2} - 1^{3/2}\right)(\ln 4 - \ln 1) \\ &= \frac{26\ln 4}{3} = \frac{52\ln 2}{3} \text{ coulombs} \end{aligned}$$

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3. [2350/041923 (28 pts)] The depth of a lake is described by the function

$$s(x,y) = \frac{-1}{1 + (x/2)^2 + (y/2)^2}$$

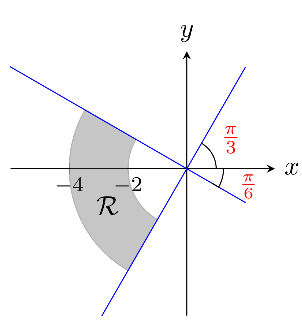
where all measurements are in kilometers. The lake's center is at the origin of a Cartesian coordinate system whose positive y-axis points north and positive x-axis points east with the lake's surface at $z = 0$. In order to conduct some underwater experiments, your lab supervisor needs to know the average depth of that part of the lake lying to the west of the lake's center, bounded by the lines $y = \sqrt{3}x$ and $y = -x/\sqrt{3}$ and the circles of radius 2 and 4 kilometers from the center of the lake.

(a) [8 pts] Make a sketch in the plane $z = 0$ of that part of the lake whose average depth you are required to find and describe it using appropriate inequalities. Be sure to label important points in your sketch.

(b) [20 pts] What number should you report to the lab supervisor? Be sure to include the appropriate units.

SOLUTION:

(a) Sketch of the lake's region of interest, $\mathcal{R}$. It is given by the polar rectangle $\left\{(r,\theta)\,\middle|\,2 \le r \le 4, \frac{5\pi}{6} \le \theta \le \frac{4\pi}{3}\right\}$.



The average depth is

$$s_{\text{avg}} = \frac{\displaystyle\iint_{\mathcal{R}} s(x,y)\,\mathrm{d}A}{\displaystyle\iint_{\mathcal{R}} \mathrm{d}A}$$

and polar coordinates are clearly going to simplify things. With that said,

$$\begin{aligned}
\iint_{\mathcal{R}} s(x,y)\,\mathrm{d}A &= \int_{5\pi/6}^{4\pi/3}\int_2^4 \frac{-1}{1+\frac{r^2}{4}}\,r\,\mathrm{d}r\,\mathrm{d}\theta \qquad \left(u = 1+\frac{r^2}{4}\right)\\
&= -2\int_{5\pi/6}^{4\pi/3}\int_2^5 \frac{1}{u}\,\mathrm{d}u\,\mathrm{d}\theta\\
&= -2\left(\frac{4\pi}{3}-\frac{5\pi}{6}\right)\ln|u|\Big|_2^5 = -\pi\ln\frac{5}{2}
\end{aligned}$$

and the area of the region is

$$\begin{aligned}
\iint_{\mathcal{R}} \mathrm{d}A &= \int_{5\pi/6}^{4\pi/3}\int_2^4 r\,\mathrm{d}r\,\mathrm{d}\theta\\
&= \int_{5\pi/6}^{4\pi/3} \frac{r^2}{2}\Big|_2^4\,\mathrm{d}\theta\\
&= \int_{4\pi/3}^{5\pi/6} 6\,\mathrm{d}\theta = 3\pi
\end{aligned}$$

The average value of the depth function is $-\frac{1}{3}\ln\frac{5}{2}$. You should tell the lab supervisor that the average depth of the lake is $\frac{1}{3}\ln\frac{5}{2}$ kilometers. (Either expression is correct).

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4. [2350/041923 (20 pts)] Let $\mathcal{S}$ be a thin lamina consisting of the portion of $2z = \sqrt{1+x^2+y^2}$ lying inside the cylinder of radius 2 aligned along the z–axis. Find the mass (including units) of $\mathcal{S}$ if the density (g/cm^2) of the material that comprises $\mathcal{S}$ is

$$\delta(x,y,z) = \frac{z(x+5)}{\sqrt{5x^2+5y^2+4}}$$

SOLUTION:

To find the mass, we compute the integral of the density over the surface, $\mathcal{S}$, which is the top branch of a hyperboloid of two sheets. We project the surface onto the xy–plane so that $\mathbf{p} = \mathbf{k}$ and the region of integration is $x^2+y^2 \le 4$ (that part of $\mathcal{S}$ inside the cylinder).

$$g(x,y,z) = \sqrt{1+x^2+y^2} - 2z \implies \nabla g = \left\langle \frac{x}{\sqrt{1+x^2+y^2}}, \frac{y}{\sqrt{1+x^2+y^2}}, -2\right\rangle \implies |\nabla g\cdot\mathbf{p}| = |-2| = 2$$

$$\implies \|\nabla g\| = \sqrt{\frac{x^2}{1+x^2+y^2}+\frac{y^2}{1+x^2+y^2}+4} = \sqrt{\frac{5x^2+5y^2+4}{1+x^2+y^2}}$$

Then, using the surface to eliminate z,

$$\begin{aligned}
\text{Mass} &= \iint_{\mathcal{S}} \delta(x,y,z)\,\mathrm{d}S = \iint_{\mathcal{R}} \delta(x,y,z)\frac{\|\nabla g\|}{|\nabla g\cdot\mathbf{p}|}\,\mathrm{d}A = \iint\limits_{x^2+y^2\le 4} \frac{z(x+5)}{\sqrt{5x^2+5y^2+4}}\sqrt{\frac{5x^2+5y^2+4}{1+x^2+y^2}}\left(\frac{1}{2}\right)\mathrm{d}A\\
&= \frac{1}{4}\iint\limits_{x^2+y^2\le 4} (x+5)\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{4}\int_0^{2\pi}\int_0^2 (r\cos\theta+5)\,r\,\mathrm{d}r\,\mathrm{d}\theta\\
&= \frac{1}{4}\left[\left(\int_0^{2\pi}\cos\theta\,\mathrm{d}\theta\right)\left(\int_0^2 r^2\,\mathrm{d}r\right) + 5\left(\int_0^{2\pi}\mathrm{d}\theta\right)\left(\int_0^2 r\,\mathrm{d}r\right)\right] = \frac{1}{4}\left[0+5(2\pi)(2)\right] = 5\pi \text{ g}
\end{aligned}$$

■

5. [2350/041923 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

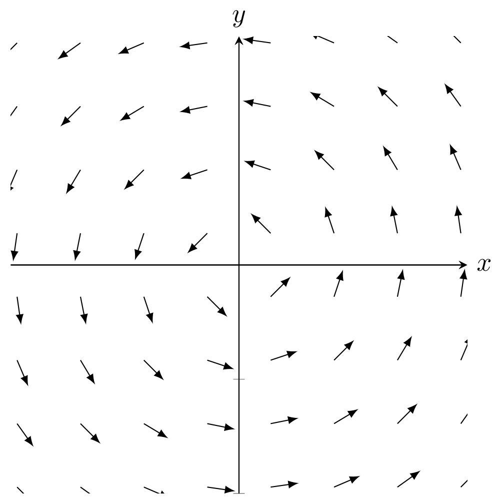
(a) Considering points only in the first octant, if $\mathbf{F} = \dfrac{1}{x}\,\mathbf{i} + \dfrac{1}{y}\,\mathbf{j} + \dfrac{1}{z}\,\mathbf{k}$ and $f = \ln(xyz)$, then $\mathbf{F}$ is conservative.

(b) $\mathbf{G} = x\,\mathbf{i} + xy\,\mathbf{j} - \mathbf{k}$ is irrotational.

(c) If $\mathbf{H} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and $\phi(x, y, z)$ is a function with continuous partial derivatives of all orders, then $\nabla \cdot (\phi\mathbf{H}) = 3\phi + \mathbf{H} \cdot \nabla\phi$.

(d) If $f(x, y, z) = xyz^{-1}$, then the divergence of the gradient of f is 0 for all $x, y, z > 0$.

(e) The vector field $\mathbf{V} = \dfrac{y}{\sqrt{x^2+y^2}}\,\mathbf{i} + \dfrac{x}{\sqrt{x^2+y^2}}\,\mathbf{j}$ is shown in the accompanying figure.



SOLUTION:

(a) **TRUE** $\mathbf{F} = \nabla f$

(b) **FALSE**

$$\nabla \times \mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & xy & -1 \end{vmatrix} = y\,\mathbf{k} \neq \mathbf{0}$$

(c) **TRUE**

$$\nabla \cdot (\phi\mathbf{H}) = \phi\nabla \cdot \mathbf{H} + \mathbf{H} \cdot \nabla\phi = 3\phi + \mathbf{H} \cdot \nabla\phi$$

(d) **FALSE**

$$\nabla \cdot (\nabla f) = \nabla^2 f = \nabla \cdot \left\langle yz^{-1}, xz^{-1}, -xyz^{-2} \right\rangle = 2xyz^{-3} \neq 0 \text{ for all } x, y, z > 0$$

(e) **FALSE** The vector field shown in the figure is $\mathbf{V} = \dfrac{-y}{\sqrt{x^2+y^2}}\,\mathrm{i} + \dfrac{x}{\sqrt{x^2+y^2}}\,\mathrm{j}$