1. [2350/031523 (26 pts)] The following parts (a), (b), and (c) are not related.

   (a) [14 pts] Let $F(x, y, z) = \left(1 - x^2\right)^{1/2} e^{y^2+z^2}$.

      i. [4 pts] Find the domain of the function.

      ii. [4 pts] Does the level surface $F(x, y, z) = -1$ exist? (answer YES or NO). If yes, find it. If no, explain why not.

      iii. [6 pts] Find the direction vector of the normal line to the surface $F(x, y, z) = \frac{\sqrt{2}e}{2}$ at the point $\left(\frac{1}{\sqrt{2}}, 0, 1\right)$.

   (b) [5 pts] Evaluate $\displaystyle\lim_{(x,y)\to(2,-2)} \frac{x^2 - y^2}{x + y}$ or explain why the limit does not exist.

   (c) [7 pts] Find the rate of change of w with respect to v at the point $(u, v) = (-1, 2)$ if $w = xy + \ln z$, $x = u^2/v$, $y = u + v$ and $z = \cos u$.

SOLUTION:

(a) i. There are no restrictions on y or z. Because of the square root we need $1 - x^2 \geq 0 \implies |x| \leq 1$. Thus, the domain is

$$\left\{(x, y, z) \in \mathbb{R}^3 \,\middle|\, |x| \leq 1\right\}$$

ii. Since exponentials and square roots are never negative, the range of the function is $[0, \infty)$. Therefore, NO, the level surface where $F(x, y, z) = -1$ does not exist.

iii. The direction vector is given by the gradient vector at the point.

$$F_x(x, y, z) = \frac{1}{2}\left(1 - x^2\right)^{-1/2}(-2x)e^{y^2+z^2} \implies F_x\left(\tfrac{1}{\sqrt{2}}, 0, 1\right) = \frac{1}{2}\left(\frac{1}{2}\right)^{-1/2}\left(-\frac{2}{\sqrt{2}}\right)e^{0^2+1^2} = -e$$

$$F_y(x, y, z) = 2y\left(1 - x^2\right)^{1/2} e^{y^2+z^2} \implies F_y\left(\tfrac{1}{\sqrt{2}}, 0, 1\right) = 0$$

$$F_z(x, y, z) = 2z\left(1 - x^2\right)^{1/2} e^{y^2+z^2} \implies F_z\left(\tfrac{1}{\sqrt{2}}, 0, 1\right) = 2(1)\left(\tfrac{1}{2}\right)^{1/2} e^{0^2+1^2} = \sqrt{2}e$$

The direction vector of the normal line to the surface is $\nabla F\left(\frac{1}{\sqrt{2}}, 0, 1\right) = \left(-e, 0, \sqrt{2}e\right)$

(b)

$$\lim_{(x,y)\to(2,-2)} \frac{x^2 - y^2}{x + y} = \lim_{(x,y)\to(2,-2)} \frac{(x + y)(x - y)}{x + y} = \lim_{(x,y)\to(2,-2)} (x - y) = 2 - (-2) = 4$$

(c) $w = w(x(u, v), y(u, v), z(u, v))$ so that

$$\begin{aligned}
\frac{\partial w}{\partial v} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v} \\
&= y\left(-\frac{u^2}{v^2}\right) + x(1) + \frac{1}{z}(0) = (u + v)\left(-\frac{u^2}{v^2}\right) + \frac{u^2}{v}
\end{aligned}$$

$$\implies \left.\frac{\partial w}{\partial v}\right|_{(-1,2)} = (-1 + 2)\left(-\frac{(-1)^2}{2^2}\right) + \frac{(-1)^2}{2} = (1)\left(-\frac{1}{4}\right) + \frac{1}{2} = \frac{1}{4}$$

■

2. [2350/031523 (21 pts)] The following parts (a) and (b) are not related.

   (a) [8 pts] The magnetic field in the first octant of $\mathbb{R}^3$ is given by $B = \ln(xyz)$. To recharge your spaceship's fuel cells in the most efficient way possible, you want to guide your ship in the direction that produces the greatest rate of change of the magnetic field with respect to distance. When you are at the point $(1, 1, 2)$, find the direction you should aim your ship and determine the corresponding rate of change of the magnetic field.

   (b) [13 pts] A rectangular metal plate occupies the region $|x| \leq 4, |y| \leq 6$. Its thickness is given by the continuous function $h(x, y)$. Use the following information about $h(x, y)$ to answer the given questions below.

$$\frac{\partial h}{\partial x} = -2xe^y \qquad \frac{\partial h}{\partial y} = e^y(2y + y^2 - x^2)$$

$$\frac{\partial^2 h}{\partial x^2} = -2e^y \qquad \frac{\partial^2 h}{\partial y^2} = e^y\left(2 + 4y + y^2 - x^2\right) \qquad \frac{\partial^2 h}{\partial x \partial y} = -2xe^y$$

i. [8 pts] Are there any points in the plate that are locally thicker or thinner than their nearby surroundings? If so, where are they? If not, explain why not.

ii. [5 pts] Is there a thinnest part of the plate? Do not find it, simply answer YES or NO and give a brief explanation justifying your answer.

SOLUTION:

(a) We need the gradient of the magnetic field.

$$\nabla B = \left\langle \frac{1}{xyz}(yz), \frac{1}{xyz}(xz), \frac{1}{xyz}(xy) \right\rangle = \left\langle \frac{1}{x}, \frac{1}{y}, \frac{1}{z} \right\rangle$$

The maximum rate of change of the magnetic field occurs in the direction of the gradient so the ship should be aimed in the direction

$$\nabla B(1,1,2) = \left\langle 1, 1, \frac{1}{2} \right\rangle = \mathbf{i} + \mathbf{j} + \frac{1}{2}\,\mathbf{k}$$

and the maximum rate of change of the magnetic field will be given by

$$\|\nabla B(1,1,2)\| = \sqrt{1^2 + 1^2 + \left(\frac{1}{2}\right)^2} = \frac{3}{2}$$

(b) i. We need to find and classify the critical points.

$$h_x = -2xe^y = 0 \implies x = 0$$

$$h_y = e^y\left(2y + y^2 - x^2\right) = 0 \implies 2y + y^2 = 0 \quad (\text{since } x = 0) \implies y = 0, -2$$

Critical points are $(0,0), (0,-2)$. Now apply the Second Derivatives Test.

$$D(0,0) = h_{xx}(0,0)h_{yy}(0,0) - [h_{xy}(0,0)]^2 = (-2)(2) - 0^2 = -4 < 0 \implies (0,0) \text{ is a saddle point}$$

$$D(0,-2) = h_{xx}(0,-2)h_{yy}(0,-2) - [h_{xy}(0,-2)]^2 = \left(-2e^{-2}\right)\left(-2e^{-2}\right) - 0^2 = 4e^{-4} > 0$$

$$\text{and } h_{xx}(0,-2) = -2e^{-2} < 0 \implies h(0,-2) \text{ is a local maximum}$$

The thickness is a local maximum at $(0,-2)$ so there is a point that is locally thicker than its nearby surroundings. There are no points in the plate that are locally thinner than their surroundings.

ii. YES. The thickness is a continuous function and the plate is a closed, bounded region so the Extreme Value Theorem applies. Since the interior critical points are a saddle and a local maximum, the thinnest part of the plate will be on the boundary.

■

3. [2350/031523 (25 pts)] Let $g(x,y) = \cos(\pi xy) + xy^2$.

(a) [8 pts] In what direction(s) will you have to move to follow a level curve at the point $(1,1)$? Express your answer(s) as unit vector(s).

(b) [7 pts] At the point $(1,1)$, is g more sensitive to small changes in x or to small changes in y? Explain using differentials.

(c) [10 pts] Suppose $g(x,y)$ represents the temperature at the point (x,y). If you are walking along the path $\mathbf{r}(t) = \frac{1}{\sqrt{t}}\,\mathbf{i} + \frac{t^2}{16}\,\mathbf{j}$, is the temperature increasing or decreasing with respect to time as you pass through the point $\left(\frac{1}{2}, 1\right)$? At what rate?

SOLUTION:

(a) We need to find vector(s) $\mathbf{u} = \langle u_1, u_2 \rangle$ in which the directional derivative evaluated at the point $(1,1)$ vanishes. The directional derivative is

$$\mathrm{D}_{\mathbf{u}}g(x,y) = \frac{\mathrm{d}g}{\mathrm{d}s} = \nabla g(x,y) \cdot \mathbf{u} = \left\langle -\pi y \sin(\pi xy) + y^2, -\pi x \sin(\pi xy) + 2xy \right\rangle \cdot \langle u_1, u_2 \rangle$$

so that

$$\mathrm{D}_{\mathbf{u}}g(1,1) = \left.\frac{\mathrm{d}g}{\mathrm{d}s}\right|_{(1,1)} = \nabla g(1,1) \cdot \mathbf{u} = \left\langle -\pi(1)\sin(\pi) + 1^2, -\pi(1)\sin(\pi) + 2(1)(1) \right\rangle \cdot \langle u_1, u_2 \rangle$$

$$= \langle 1, 2 \rangle \cdot \langle u_1, u_2 \rangle = u_1 + 2u_2 = 0 \implies u_1 = -2u_2$$

Thus, for example, $u_2 = 1, u_1 = -2 \implies \mathbf{u} = -2\,\mathbf{i} + \mathbf{j}$ or $u_2 = -1, u_1 = 2 \implies \mathbf{u} = 2\,\mathbf{i} - \mathbf{j}$. The unit vectors we seek are $\pm\frac{\sqrt{5}}{5}\left(2\,\mathbf{i} - \mathbf{j}\right)$.

(b)

$$\mathrm{d}g = \frac{\partial g}{\partial x}\,\mathrm{d}x + \frac{\partial g}{\partial y}\,\mathrm{d}y = \left[-\pi y\sin(\pi xy) + y^2\right]\mathrm{d}x + \left[-\pi x\sin(\pi xy) + 2xy\right]\mathrm{d}y$$

At $(1, 1)$ we have $\mathrm{d}g = 1\,\mathrm{d}x + 2\,\mathrm{d}y$ so that g is more sensitive to small changes in y.

(c) You arrive at the point $\left(\frac{1}{2}, 1\right)$ when $t = 4$ and the rate of change of temperature with respect to time is given by

$$\frac{\mathrm{d}g}{\mathrm{d}t} = \nabla g \cdot \mathbf{r}'(t) = \langle -\pi y\sin(\pi xy) + y^2, -\pi x\sin(\pi xy) + 2xy\rangle \cdot \left\langle -\frac{1}{2}t^{-3/2}, \frac{t}{8}\right\rangle$$

$$\implies \left.\frac{\mathrm{d}g}{\mathrm{d}t}\right|_{t=4} = \langle -\pi(1)\sin(\pi/2) + 1^2, -\pi(1/2)\sin(\pi/2) + 2(1/2)(1)\rangle \cdot \left\langle -\frac{1}{2}4^{-3/2}, \frac{4}{8}\right\rangle$$

$$= \left\langle 1-\pi, 1-\frac{\pi}{2}\right\rangle \cdot \left\langle -\frac{1}{16}, \frac{1}{2}\right\rangle = -\frac{1}{16} + \frac{\pi}{16} + \frac{8}{16} - \frac{4\pi}{16} = \frac{1}{16}(7-3\pi)$$

Since this is negative, the temperature is decreasing at a rate of $\left|\frac{1}{16}(7-3\pi)\right|$

■

4. [2350/031523 (16 pts)] A skateboard park has a new track in the form of the hyperbola $\frac{1}{4}x^2 - y^2 = 12$. You are standing at the point $(x, y) = (0, 10)$ watching your friends skate on the track. Using Lagrange Multipliers, determine the closest your friends will get to you. Where will they be when this occurs?

SOLUTION:

The distance from you to an arbitrary point, (x, y), on the track is $d(x, y) = \sqrt{x^2 + (y-10)^2}$. To simplify things, we will minimize the square of the distance function, $d^2(x, y) = f(x, y) = x^2 + (y-10)^2$. Since the arbitrary point must live on the hyperbola (your friends are skating on the track), the constraint is $g(x, y) = \frac{1}{4}x^2 - y^2 = 12$.

$$f_x = 2x \qquad g_x = \frac{1}{2}x$$
$$f_y = 2(y-10) \qquad g_y = -2y$$

We need to find the solutions of the following system of nonlinear equations

$$2x = \frac{1}{2}\lambda x \implies 4x = \lambda x \tag{1}$$
$$2(y-10) = -2\lambda y \implies y - 10 = -\lambda y \tag{2}$$
$$\frac{1}{4}x^2 - y^2 = 12 \tag{3}$$

Equation (1) is equivalent to $4x - \lambda x = x(4-\lambda) = 0 \implies x = 0, \lambda = 4$. If $x = 0$, then Eq. (3) is $-y^2 = 12$, which has no solution. Thus we must have $\lambda = 4$ in Eq. (1). Equation (2) then becomes $y - 10 = -4y \implies y = 2$. From Eq. (3) we then have

$$\frac{1}{4}x^2 - 2^2 = 12$$
$$x^2 = 64$$
$$x = \pm 8$$

The critical points are thus $(\pm 8, 2)$. These will produce the minimum distance since x and y can both approach infinity on the constraint (it is unbounded) producing an arbitrarily large distance from $(0, 10)$. Your friends will be closest to you when they are at either $(8, 2)$ or $(-8, 2)$. In either case the distance between you and them will be $\sqrt{(\pm 8)^2 + (2-10)^2} = 8\sqrt{2}$. ■

5. [2350/031523 (12 pts)] The following parts (a) and (b) are not related.

(a) [7 pts] Find the second order/quadratic Taylor approximation of $f(x, y) = e^{-(x^2+2y)}$ centered at the point $\left(1, -\frac{1}{2}\right)$. Do not simplify your answer.

(b) [5 pts] Suppose you have the following information about a function, $g(x, y)$:

$$g_{xx}(x, y) = \frac{1}{x^2+y^2+2}, \qquad g_{xy} = \frac{-3}{x^2+y^2+2}, \qquad g_{yy} = \frac{20x}{x^2+y^2+2}$$

Using this information, find an upper bound on the error in the first order/linear Taylor approximation for $g(x, y)$ centered at the origin if $|x| \leq 0.1$ and $|y| \leq 0.3$. Do not find the Taylor approximation (you can't actually).

SOLUTION:

(a)

$$
\begin{aligned}
f\left(1,-\tfrac{1}{2}\right) &= 1\\
f_x(x,y) &= -2xe^{-(x^2+2y)} \implies f_x\left(1,-\tfrac{1}{2}\right) = -2\\
f_{xx}(x,y) &= -2e^{-(x^2+2y)}\left(-2x^2+1\right) \implies f_{xx}\left(1,-\tfrac{1}{2}\right) = 2\\
f_{xy}(x,y) &= 4xe^{-(x^2+2y)} \implies f_{xy}\left(1,-\tfrac{1}{2}\right) = 4\\
f_y(x,y) &= -2e^{-(x^2+2y)} \implies f_y\left(1,-\tfrac{1}{2}\right) = -2\\
f_{yy}(x,y) &= 4e^{-(x^2+2y)} \implies f_{yy}\left(1,-\tfrac{1}{2}\right) = 4\\
\implies T_2(x,y) &= 1-2(x-1)-2\left(y+\tfrac{1}{2}\right)+\frac{1}{2!}\left[2(x-1)^2+2(4)(x-1)\left(y+\tfrac{1}{2}\right)+4\left(y+\tfrac{1}{2}\right)^2\right]
\end{aligned}
$$

(b) We need to maximize $|g_{xx}|$, $|g_{xy}|$, and $|g_{yy}|$ on the rectangle $-0.1 \le x \le 0.1$, $-0.3 \le y \le 0.3$.

$$
\begin{aligned}
|g_{xx}| &\le \frac{1}{0^2+0^2+2} = \frac{1}{2}\\
|g_{xy}| &\le \frac{3}{0^2+0^2+2} = \frac{3}{2}\\
|g_{yy}| &\le \frac{20(0.1)}{(0.1)^2+0^2+2} = \frac{200}{201} < 1
\end{aligned}
$$

So choose $M = \max_{\substack{|x|\le 0.1\\|y|\le 0.3}} \{|g_{xx}|, |g_{xy}|, |g_{yy}|\} = \frac{3}{2}$ yielding

$$
|E(x,y)| \le \frac{3/2}{2}(0.1+0.3)^2 = \frac{3}{4}\left(\frac{16}{100}\right) = \frac{3}{25}
$$

$\blacksquare$