1. [2350/021523 (15 pts)] Han Solo and Chewbacca are piloting the Millennium Falcon along the path given by

$$\mathbf{r}(t) = \cos(\pi t)\,\mathbf{i} + \sin(\pi t)\,\mathbf{j} + (4t - t^2)\,\mathbf{k}, \;\; t \geq 0$$

When $t = 3$, Chewy fires a laser beam from the front of the ship that travels forward in a straight line. Find the coordinates of the point where the laser beam penetrates the xy-plane.

SOLUTION:

The laser beam follows the tangent line. To find the equation of the tangent line, we need a point on the tangent line and its direction. When $t = 3$, the fighter is at $\mathbf{r}(3) = \cos 3\pi\,\mathbf{i} + \sin 3\pi\,\mathbf{j} + \left[4(3) - 3^2\right]\,\mathbf{k} = -\mathbf{i} + 3\,\mathbf{k}$, so a point on the curve and on the tangent line is $(-1, 0, 3)$.

The velocity vector is $\mathbf{v}(t) = \mathbf{r}'(t) = -\pi\sin(\pi t)\,\mathbf{i} + \pi\cos(\pi t)\,\mathbf{j} + (4 - 2t))\,\mathbf{k}$ and when $t = 3$ is $\mathbf{v}(3) = -\pi\sin 3\pi\,\mathbf{i} + \pi\cos 3\pi\,\mathbf{j} + [4 - 2(3)]\,\mathbf{k} = -\pi\,\mathbf{j} - 2\,\mathbf{k}$. This gives the direction of the tangent line, so its equation is

$$\mathbf{L}(s) = \langle -1, 0, 3\rangle + s\langle 0, -\pi, -2\rangle = \langle -1, -\pi s, 3 - 2s\rangle, \quad -\infty < s < \infty$$

This line intersects the xy-plane when its z-coordinate vanishes, which occurs if $3 - 2s = 0 \implies s = \frac{3}{2}$. The laser intersects the xy-plane at the point $(-1, -\frac{3\pi}{2}, 0)$. ■

2. [2350/021523 (12 pts)] Consider the equation $-\frac{1}{4}z^2 - 8y^2 + x^2 + 2z - 20 = 0$

   (a) [4 pts] Name the surface, providing justification for your answer.

   (b) [4 pts] Does the surface intersect the yz-plane? Justify your answer.

   (c) [4 pts] Name the conic section of the trace when $x = -\sqrt{32}$, providing justification for your answer.

SOLUTION:

(a) Complete the square.

$$x^2 - 8y^2 - \frac{1}{4}(z^2 - 8z + 16 - 16) - 20 = 0$$

$$x^2 - 8y^2 - \frac{1}{4}(z - 4)^2 = 20 - 4 = 16$$

$$-\frac{x^2}{16} + \frac{y^2}{2} + \frac{(z-4)^2}{64} = -1 \quad \text{or} \quad \frac{x^2}{16} - \frac{y^2}{2} - \frac{(z-4)^2}{64} = 1 \quad \text{or} \quad -\left(\frac{x}{4}\right)^2 + \left(\frac{y}{\sqrt{2}}\right)^2 + \left(\frac{z-4}{8}\right)^2 = -1$$

This is a hyperboloid of two sheets.

(b) Intersecting the yz-plane means that $x = 0$. Substituting this into the equation gives the result

$$\frac{y^2}{2} + \frac{(z-4)^2}{64} = \left(\frac{y}{\sqrt{2}}\right)^2 + \left(\frac{z-4}{8}\right)^2 = -1,$$

an equation with no solution (sum of two squares is never negative). Thus, the surface does not intersect the yz-plane. Alternatively, using the original equation we arrive at the same conclusion.

$$-\frac{1}{4}z^2 - 8y^2 + 2z - 20 = 0$$

$$z^2 + 32y^2 - 8z + 80 = 0$$

$$z^2 - 8z + 16 + 32y^2 + 80 - 16 = 0$$

$$(z-4)^2 + 32y^2 = -64$$

(c) To find the trace, substitute $x = -\sqrt{32}$ into the equation. The result is

$$-\frac{(-\sqrt{32})^2}{16} + \frac{y^2}{2} + \frac{(z-4)^2}{64} = -1 \implies \left(\frac{y}{\sqrt{2}}\right)^2 + \left(\frac{z-4}{8}\right)^2 = 1$$

which is an ellipse. Alternatively, using the original equation we arrive at the same result.

$$-\frac{1}{4}z^2 - 8y^2 + \left(-\sqrt{32}\right)^2 + 2z - 20 = 0$$

$$z^2 + 32y^2 - 128 - 8z + 80 = 0$$

$$z^2 - 8z + 16 + 32y^2 - 48 - 16 = 0$$

$$(z-4)^2 + 32y^2 = 64$$

$$\left(\frac{z-4}{8}\right)^2 + \left(\frac{y}{\sqrt{2}}\right)^2 = 1$$

■

3. [2350/021523 (15 pts)] Find the equation of the plane that is perpendicular to the plane $2z = 5x + 4y$ and contains the line with symmetric equations $-x = \frac{y+2}{5} = \frac{z-5}{-4}$. Write your final answer in the form $ax + by + cz = d$.

SOLUTION:

Let P be the plane whose equation we seek. We need a point in P and its normal vector, $\mathbf{n}$. Begin by noting that the line has parametric equations

$$x = -t \quad y = -2 + 5t \quad z = 5 - 4t$$

giving a direction vector of $\mathbf{v}_1 = -\mathbf{i} + 5\,\mathbf{j} - 4\,\mathbf{k}$ which is parallel to P. A point in P is on the given line, say when $t = 0$, giving $(0, -2, 5)$.

The normal vector to the given plane is $\mathbf{v}_2 = 5\,\mathbf{i} + 4\,\mathbf{j} - 2\,\mathbf{k}$, which is parallel to P since the two planes are perpendicular. The cross product of $\mathbf{v}_1$ and $\mathbf{v}_2$ will be normal to both of these vectors and thus provide the normal, $\mathbf{n}$, to P.

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 5 & -4 \\ 5 & 4 & -2 \end{vmatrix} = 6\,\mathbf{i} - 22\,\mathbf{j} - 29\,\mathbf{k}$$

Thus

$$6(x-0) - 22(y+2) - 29(z-5) = 0 \implies 6x - 22y - 44 - 29z + 145 = 0 \implies 6x - 22y - 29z = -101$$

■

4. [2350/021523 (46 pts)] A scorpion is crawling on a shelf located 2 meters above the floor in a room. Its path is given by

$$\mathbf{r}(t) = \frac{t^3}{3}\mathbf{i} + \frac{t^2}{2}\mathbf{j} + 2\,\mathbf{k} \quad t \geq 0,$$

with distances measured in meters. Answer **ALL** of the following questions for $t = 1$ second.

(a) [3 pts] Where is the scorpion?

(b) [4 pts] How fast is the scorpion crawling? (Include units in your answer)

(c) [8 pts] Briefly describe in words (*i.e.* DO NOT COMPUTE) what the following two quantities represent physically in terms of the scorpion's path:

i. $\sqrt{\mathbf{r}(t) \cdot \mathbf{r}(t)}$ ii. $\int_0^t \sqrt{\mathbf{r}'(u) \cdot \mathbf{r}'(u)}\,\mathrm{d}u$

(d) [5 pts] Find the scorpion's unit tangent vector, $\mathbf{T}$.

(e) [5 pts] Find the unit normal to the scorpion's path, which for this curve can be accomplished by computing $\mathbf{T} \times \mathbf{k}$.

(f) [5 pts] Find the binormal vector to the scorpion's path by calculating $\mathbf{T} \times \mathbf{N}$.

(g) [4 pts] Find the equation of the scorpion's osculating plane.

(h) [4 pts] In your bluebook, draw an xyz-coordinate system according to the right hand rule such that the positive z-axis is perpendicular to and out of your paper and your paper is the plane $z = 2$. Then draw $\mathbf{T}(1)$ and $\mathbf{N}(1)$ at the appropriately labeled point. Be sure to label the vectors correctly.

(i) [4 pts] How fast is the scorpion's speed changing? (Include units in your answer)

(j) [4 pts] Does the scorpion's acceleration possess a component normal to its path? If so, find its magnitude, including units in your answer. If not, explain why not.

SOLUTION:

(a) This is just the position vector evaluated at $t = 1$.

$$\mathbf{r}(1) = \frac{1}{3}\,\mathbf{i} + \frac{1}{2}\,\mathbf{j} + 2\,\mathbf{k} \text{ or } \left(\frac{1}{3}, \frac{1}{2}, 2\right)$$

(b) Need to find the speed, $\|\mathbf{v}(1)\|$. We have

$$\mathbf{v}(t) = \mathbf{r}'(t) = t^2\,\mathbf{i} + t\,\mathbf{j} \implies \mathbf{v}(1) = \mathbf{i} + \mathbf{j} \implies \|\mathbf{v}(1)\| = \sqrt{2} \text{ m/s}$$

(c) i. $\sqrt{\mathbf{r}(1)\cdot\mathbf{r}(1)} = \|\mathbf{r}(1)\|$, which is the distance the scorpion is from the origin after one second.

ii. $\displaystyle\int_0^1 \sqrt{\mathbf{r}'(u)\cdot\mathbf{r}'(u)}\,\mathrm{d}u = \int_0^1 \|\mathbf{r}'(u)\|\,\mathrm{d}u$, which gives the distance the scorpion has actually crawled during the first second, that is, the arc length of the scorpion's path over the first second.

(d) From part (b), $\|\mathbf{r}'(t)\| = \sqrt{t^4+t^2} = \sqrt{t^2(t^2+1)} = |t|\sqrt{t^2+1} = t\sqrt{t^2+1}$ since $t \geq 0$.

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{t^2\,\mathbf{i} + t\,\mathbf{j}}{t\sqrt{t^2+1}} = \frac{t}{\sqrt{t^2+1}}\,\mathbf{i} + \frac{1}{\sqrt{t^2+1}}\,\mathbf{j} \implies \mathbf{T}(1) = \frac{1}{\sqrt{2}}\,\mathbf{i} + \frac{1}{\sqrt{2}}\,\mathbf{j}$$

Alternatively, we have all of the information we need from part (b)

$$\mathbf{T}(1) = \frac{\mathbf{v}(1)}{\|\mathbf{v}(1)\|} = \frac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}$$

(e)

$$\mathbf{N}(1) = \mathbf{T}(1)\times\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{1}{\sqrt{2}}\,\mathbf{i} - \frac{1}{\sqrt{2}}\,\mathbf{j}$$

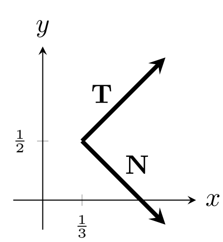
(f)

$$\mathbf{B}(1) = \mathbf{T}(1)\times\mathbf{N}(1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{vmatrix} = -\mathbf{k}$$

(g) Since $\mathbf{T}$ and $\mathbf{N}$ have no $\mathbf{k}$-component, the osculating plane is parallel to the xy-plane. Since the trajectory (path) includes the component $2\,\mathbf{k}$, the osculating plane's equation is $z = 2$. Alternatively, the binormal vector $(-\mathbf{k})$ can serve as a normal vector to the osculating plane. A point in the osculating plane is $\left(\frac{1}{3}, \frac{1}{2}, 2\right)$. Then

$$0\left(x - \frac{1}{3}\right) + 0\left(y - \frac{1}{2}\right) + -1(z-2) = 0 \implies z = 2$$

(h) The following figure shows $\mathbf{T}$ and $\mathbf{N}$. Note that $\mathbf{B}$ is pointing into the page in the figure.



(i) We will need the acceleration vector $\mathbf{a}(t) = \mathbf{r}''(t) = 2t\,\mathbf{i} + \mathbf{j} \implies \mathbf{a}(1) = 2\,\mathbf{i} + \mathbf{j}$. The speed change is simply the tangential component of the acceleration.

$$a_T(1) = \frac{\mathbf{r}'(1)\cdot\mathbf{r}''(1)}{\|\mathbf{r}'(1)\|} = \frac{(\mathbf{i}+\mathbf{j})\cdot(2\,\mathbf{i}+\mathbf{j})}{\sqrt{2}} = \frac{3}{\sqrt{2}} \text{ m/s}^2$$

Alternatively,

$$a_T(t) = \frac{\mathrm{d}\|\mathbf{v}(t)\|}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\sqrt{t^4+t^2} = \frac{4t^3+2t}{2\sqrt{t^4+t^2}}$$

$$a_T(1) = \frac{3}{\sqrt{2}} \text{ m/s}^2$$

(j) Yes.

$$a_N(1) = \frac{\|\mathbf{r}'(1) \times \mathbf{r}''(1)\|}{\|\mathbf{r}'(1)\|} = \frac{1}{\sqrt{2}} \left\| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{vmatrix} \right\| = \frac{1}{\sqrt{2}} \| -\mathbf{k} \| = \frac{1}{\sqrt{2}} \text{ m/s}^2$$

Alternatively,

$$\mathbf{a}(t) = \mathbf{r}''(t) = 2t\,\mathbf{i} + \mathbf{j} \implies \|\mathbf{a}(t)\| = \sqrt{4t^2 + 1}$$

$$a_N(1) = \sqrt{\|\mathbf{a}(1)\|^2 - a_T(1)^2}$$

$$= \sqrt{\left(\sqrt{4(1)^2 + 1}\right)^2 - \left(\frac{3}{\sqrt{2}}\right)^2} = \sqrt{5 - \frac{9}{2}} = \frac{1}{\sqrt{2}} \text{ m/s}^2$$

Since this is nonzero, there is an acceleration normal to the scorpion's path.

■

5. [2350/021523 (12 pts)] Consider the force given by $\mathbf{F} = 2\,\mathbf{i} + 7\,\mathbf{j} + 3\,\mathbf{k}$ Newtons.

(a) [6 pts] Suppose you are moving in a straight line in the direction of $\mathbf{v} = 6\,\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}$. Calculate the work done by the force if you continue moving along the line a total distance of 6 meters.

(b) [6 pts] Consider a beam mounted at the point P that can rotate around that point. The force, $\mathbf{F}$, is applied to the beam at an angle of 30 degrees, resulting in a torque of magnitude 30 Newton-m. How far from P was the force applied?

SOLUTION:

(a) We begin by finding the displacement vector, $\mathbf{D}$.

$$\mathbf{D} = 6\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{6}{\sqrt{6^2 + 2^2 + 3^2}}(6\,\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}) = 6\left(\frac{6}{7}\,\mathbf{i} + \frac{2}{7}\,\mathbf{j} + \frac{3}{7}\,\mathbf{k}\right)$$

Then

$$\text{Work} = \mathbf{F} \cdot \mathbf{D} = (2\,\mathbf{i} + 7\,\mathbf{j} + 3\,\mathbf{k}) \cdot 6\left(\frac{6}{7}\,\mathbf{i} + \frac{2}{7}\,\mathbf{j} + \frac{3}{7}\,\mathbf{k}\right) = \frac{6}{7}(12 + 14 + 9) = 30\,\text{Newton-m} = 30\,\text{Joules}$$

(b)

$$\|\mathbf{r}\| = \frac{\|\boldsymbol{\tau}\|}{\|\mathbf{F}\| \sin 30^\circ} = \frac{30}{\sqrt{2^2 + 7^2 + 3^2}(1/2)} = \frac{60}{\sqrt{62}} \text{ m} = \frac{30\sqrt{62}}{31} \text{ m}$$

■