APPM 2350—Final Exam

Saturday April 30th, 10:30am-1pm 2022

This exam has 5 problems. Please start each new problem at the top of a new page in your blue book. Show all your work in your blue book and simplify your answers. Answers with missing or insufficient justification will receive no points. You are allowed one 8.5×11-in page of notes (TWO sided). You may NOT use a calculator, smartphone, smartwatch, the Internet or any other electronic device.

Problem 1 (30 points) The following questions are not related:

(a) Find the value(s) of a and b where the function

$$I(a,b) = \int_a^b (-2x^2 + 10x - 12)\,dx$$

has a local maximum. Be sure to support your answer using Calculus 3 concepts.

<u>**SOLUTION:**</u>

To support our answer to this problem using Calculus 3 concepts, we must find and classify all critical points of $I(a,b)$. We can check our answer by graphing the integrand (it's a downward opening parabola), but only using the graph of the integrand to answer this question is not sufficient justification for full credit on this problem. This problem is asking us to connect what we may have learned in Calc 1 to the Calc 3 concept of finding the local maxima of functions of 2 variables.

Local max/min/saddles occur when $\nabla I = \vec{0}$.

Using the Fundamental Theorem of Calculus:

$$I_b = \frac{\partial}{\partial b}\int_a^b (-2x^2 + 10x - 12)\,dx = -2b^2 + 10b - 12$$

$$I_a = \frac{\partial}{\partial a}\int_a^b (-2x^2 + 10x - 12)\,dx = -(-2a^2 + 10a - 12)$$

$$\nabla I = \vec{0} \tag{1}$$

$$\implies \left\langle 2a^2 - 10a + 12, -2b^2 + 10b - 12\right\rangle = \langle 0, 0\rangle \tag{2}$$

Resulting in the system of equations:

$$2a^2 - 10a + 12 = 0$$
$$-2b^2 + 10b - 12 = 0$$

.

which we can factor:

$$2(a-3)(a-2) = 0$$
$$-2(b-3)(b-2) = 0$$

The values $a = 2, 3$ solve the first equation and $b = 2, 3$ solve the second equation.

The possible local maximums occur at: $(2,2)$, $(2,3)$, $(3,2)$, and $(3,3)$.

We can find which one is a local max by using $D = I_{aa}I_{bb} - {I_{ab}}^2$.
For $I(a,b)$: $I_{aa} = 4a - 10$, $I_{bb} = -4b + 10$, and $I_{ab} = 0$.
Thus $D = (4a - 10)(10 - 4b)$

$D(2,2) < 0$ (saddle point)
$D(3,3) < 0$ (saddle point)
$D(3,2) > 0$ and $I_{aa}(3,2) > 0$, thus this is a local min
Since $D(2,3) = (4(2) - 10)(-4(3) + 10) + 0^2 = 4$ is positive and $I_{aa}(2,3) = -2$ is negative then there is a $\boxed{\text{local maximum at (a,b)=(2,3)}}$

■

(b) Consider the function

$$f(x, y, z) = x^2 y - z\cos(y)$$

Use a directional derivative to approximate how much f changes if one moves a distance 0.1 from the point $(4, 0, 3)$ straight toward the origin.

SOLUTION:

We use the formulation:

$$D_{\hat{u}} f = \nabla f \cdot \hat{u}$$

The vector pointing from $(4, 0, 3)$ to the origin $(0, 0, 0)$ is $\langle 0 - 4, 0 - 0, 0 - 3 \rangle = \langle -4, 0, -3 \rangle$. Finding the unit vector we get:

$$\hat{u} = \frac{\langle -4, 0, -3 \rangle}{\sqrt{(-4)^2 + 0^2 + (-3)^2}} = \left\langle -\frac{4}{5}, 0, -\frac{3}{5} \right\rangle.$$

The gradient of f is $\nabla f = \langle 2xy, x^2 + z\sin y, -\cos y \rangle$. So the directional derivative is:

$$D_{\hat{u}} f = \nabla f \cdot \hat{u} = \langle 2xy, x^2 + z\sin y, -\cos y \rangle \cdot \left\langle -\frac{4}{5}, 0, -\frac{3}{5} \right\rangle = -\frac{8}{5}xy + 0 + \frac{3}{5}\cos y$$

The directional derivative at $(4, 0, 3)$ is:

$$D_{\hat{u}} f(4, 0, 3) = -\frac{8}{5}(4)(0) + \frac{3}{5}\cos(0) = \frac{3}{5}.$$

So the approximate change of f in the direction of $\hat{u}$ over a distance of 0.1 is given by: $\frac{3}{5}\frac{1}{10} = \boxed{\frac{3}{50}}$.

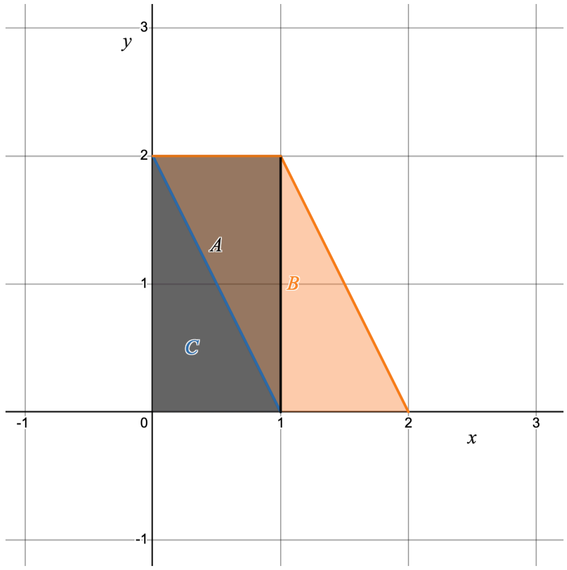
■

(c) Arrange the following three double integrals in order from least to greatest **and explain/justify your reasoning**:

$$\int_0^2 \int_0^1 e^{x^2+y^2}\, dx\, dy, \qquad \int_0^2 \int_0^{2-\frac{y}{2}} e^{x^2+y^2}\, dx\, dy, \qquad \int_0^2 \int_0^{1-\frac{y}{2}} e^{x^2+y^2}\, dx\, dy$$

SOLUTION:

We begin by noting that $e^{x^2+y^2} > 0$ for all (x, y), so we can interpret each double integral as the positive volume of a solid bounded below by the region of integration on the xy-plane and above by the surface $z = e^{x^2+y^2}$. Sketching the three regions of integration on the xy-plane we get:



Where the rectangular region, A, bounded by $x = 0$, $x = 1$, $y = 0$, and $y = 2$ in black is the region of integration for the first integral $\int_0^2 \int_0^1 e^{x^2+y^2}\, dx\, dy$. The region of integration for $\int_0^2 \int_0^{2-\frac{y}{2}} e^{x^2+y^2}\, dx\, dy$ is given by the trapezoidal region which is the sum of the orange triangular region and the rectangular region A. Finally, the triangular region, C, bounded by $x = 0$, $y = 0$, and $y = -2x + 2$ in blue is the region of integration for $\int_0^2 \int_0^{1-\frac{y}{2}} e^{x^2+y^2}\, dx\, dy$.

Since $e^{x^2+y^2} > 0$ we know that:

$$\boxed{\int_0^2 \int_0^{1-\frac{y}{2}} e^{x^2+y^2}\, dx\, dy < \int_0^2 \int_0^1 e^{x^2+y^2}\, dx\, dy < \int_0^2 \int_0^{2-\frac{y}{2}} e^{x^2+y^2}\, dx\, dy}$$

■

Problem 2 (30 points)
Given the force vector field

$$\mathbf{F}(x, y, z) = 2y\mathbf{i} + 3z\mathbf{j} - x\mathbf{k}$$

Consider the plane $\mathcal{P}$ that passes through the points $(1, 1, 3)$, $(3, 0, 1)$ and $(-2, 2, 7)$. Let $\mathcal{C}$ be any closed circular path with radius a that lies in the plane $\mathcal{P}$, oriented counterclockwise when viewed from above (that is, when viewed from the positive z-axis looking down). Note, the circular path $\mathcal{C}$ does not necessarily pass through the given points. (It is tricky to parameterize the path $\mathcal{C}$, so **don't try to parameterize** it during this exam).

(a) Without parameterizing the path, what is the curvature of the path $\mathcal{C}$?

<u>**SOLUTION:**</u>

Since the curve is a circle of radius a then the curvature is $\dfrac{1}{a}$.

■

(b) Without parameterizing the path, what is the unit binormal, $\hat{\mathbf{B}}$, to the path $\mathcal{C}$?

SOLUTION:

The unit binormal is defined to be $\hat{\mathbf{B}} = \hat{\mathbf{T}} \times \hat{\mathbf{N}}$ where $\hat{\mathbf{T}}$ is the unit tangent vector to the curve and $\hat{\mathbf{N}}$ is the unit normal vector to the curve. Since the circle lies within the plane, $\mathcal{P}$, we know that $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$ both lie within $\mathcal{P}$ as well. So $\hat{\mathbf{B}}$ is a unit vector that is normal to the plane. We will find $\hat{\mathbf{B}}$ by taking the cross product of any two vectors that lie in the plane, then we'll divide by the magnitude of the vector we find, and finally we will check the direction of the resulting unit vector.

Let

$$\begin{aligned}\tilde{\mathbf{v}}_{\mathbf{1}} &=< 3-1, 0-1, 1-3 > \quad (3)\\ &=< 2, -1, -2 > \quad (4)\end{aligned}$$

and

$$\begin{aligned}\tilde{\mathbf{v}}_{\mathbf{2}} &=< -2-1, 2-1, 7-3 > \quad (5)\\ &=< -3, 1, 4 > . \quad (6)\end{aligned}$$

So

$$\tilde{\mathbf{v}}_{\mathbf{1}} \times \tilde{\mathbf{v}}_{\mathbf{2}} = \begin{vmatrix} \hat{i} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & -1 & -2 \\ -3 & 1 & 4 \end{vmatrix} = [(-1)(4)-(-2)(1)]\,\hat{\mathbf{i}} - [(2)(4)-(-2)(-3)]\,\hat{\mathbf{j}} + [(2)(1)-(-1)(-3)]\,\hat{\mathbf{k}} = -2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - \hat{\mathbf{k}}.$$

Finding the unit vector we get $\dfrac{\tilde{\mathbf{v}}_{\mathbf{1}} \times \tilde{\mathbf{v}}_{\mathbf{2}}}{||\tilde{\mathbf{v}}_{\mathbf{1}} \times \tilde{\mathbf{v}}_{\mathbf{2}}||} = \dfrac{-2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 1\hat{\mathbf{k}}}{\sqrt{(-2)^2 + 2^2 + (-1)^2}} = -\dfrac{2}{3}\hat{\mathbf{i}} - \dfrac{2}{3}\hat{\mathbf{j}} - \dfrac{1}{3}\hat{\mathbf{k}}.$

The curve is oriented counterclockwise when looking from the positive z-direction. Because $\mathcal{C}$ is a circle we know that $\hat{\mathbf{N}}$ points toward the center of the circle for all points on the curve. So the direction of $\hat{\mathbf{B}}$ must have a positive $\hat{\mathbf{k}}$ component by the right hand rule for cross products. So $\hat{\mathbf{B}} = -\dfrac{\tilde{\mathbf{v}}_{\mathbf{1}} \times \tilde{\mathbf{v}}_{\mathbf{2}}}{||\tilde{\mathbf{v}}_{\mathbf{1}} \times \tilde{\mathbf{v}}_{\mathbf{2}}||} = \boxed{\dfrac{2}{3}\hat{\mathbf{i}} + \dfrac{2}{3}\hat{\mathbf{j}} + \dfrac{1}{3}\hat{\mathbf{k}}}$

■

(c) Without parameterizing the path, find the work done by $\mathbf{F}$ once around $\mathcal{C}$.

SOLUTION:

Let S be the surface that is the portion of the plane, $\mathcal{P}$, enclosed by the circle $\mathcal{C}$. Since $\tilde{\mathbf{F}}$ has continuous partial derivatives over the surface S then Stoke's Theorem applies:

$$\oint_{\mathcal{C}} \tilde{\mathbf{F}} \cdot \hat{\mathbf{T}}\, ds = \iint_{S} \left(\nabla \times \tilde{\mathbf{F}}\right) \cdot \hat{\mathbf{n}}\, dS.$$

We find $\nabla \times \tilde{\mathbf{F}}$:

$$\nabla \times \tilde{\mathbf{F}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & 3z & -x \end{vmatrix} = [0-3]\,\hat{\mathbf{i}} - [-1-0]\,\hat{\mathbf{j}} + [0-2]\,\hat{\mathbf{k}} = -3\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}.$$

Note that the outward pointing normal $\hat{\mathbf{n}}$ is $\hat{\mathbf{B}}$ found in part (b). $\hat{\mathbf{B}}$ points in the correct direction relative to the orientation of the curve. So

$$\left(\nabla \times \tilde{\mathbf{F}}\right) \cdot \hat{\mathbf{n}} = \left(\nabla \times \tilde{\mathbf{F}}\right) \cdot \hat{\mathbf{B}} = (-3\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}) \cdot \left(\frac{2}{3}\hat{\mathbf{i}} + \frac{2}{3}\hat{\mathbf{j}} + \frac{1}{3}\hat{\mathbf{k}}\right) = (-3)\left(\frac{2}{3}\right) + (1)\left(\frac{2}{3}\right) - (2)\left(\frac{1}{3}\right) = -2.$$

and we get

$$\iint_S \left(\nabla \times \tilde{\mathbf{F}}\right) \cdot \hat{\mathbf{n}}\, dS = \iint_S -2\, dS = -2 \iint_S dS = \boxed{-2\pi a^2}$$

since S is a circle with area πa^2.

■

Problem 3 (30 pts)
A velocity field is given by

$$\mathbf{F} = a(z + xe^y)\mathbf{i} + (bx^2 + cz)e^y\mathbf{j} + (\sin z + e^y)\mathbf{k}$$

(a) For what values of a, b and c will $\mathbf{F}$ be conservative? Be sure to justify your answer and double check your work.
(b) Using the values of a, b and c you found above, find the flow of $\mathbf{F}$ along the straight line path starting at $(0, 0, 0)$ and ending at $(1, 0, \pi)$ using an appropriate Calculus 3 theorem.
(c) Verify your answer in part (b) by direct computation (i.e. by evaluating a line integral).

<u>**SOLUTION:**</u>

(a) A vector field is conservative if its curl is $\vec{0}$. $\nabla \times \mathbf{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle$. Compute the derivatives:

$$\begin{array}{lll} & P_y = axe^y & P_z = a \\ Q_x = 2bxe^y & & Q_z = ce^y \\ R_x = 0 & R_y = e^y & \end{array}$$

Compute the values of $a,\ b,$ and c:

$$R_y - Q_z = (1 - c)e^y = 0 \Leftrightarrow \boxed{c = 1}$$
$$P_z - R_x = a = 0 \Leftrightarrow \boxed{a = 0}$$
$$Q_x - P_y = (2b - a)xe^y = 0 \Leftrightarrow \boxed{b = \frac{a}{2} = 0}$$

So

$$\boxed{\mathbf{F} = \langle 0,\ ze^y,\ \sin z + e^y\rangle}$$

We can check our work by finding a potential function $f(x, y, z)$ such that $\nabla f = \mathbf{F}$:

$$f_x = 0 \Leftrightarrow f(x, y, z) = g(y, z)$$
$$f_y = ze^y \Leftrightarrow f(x, y, z) = ze^y + h(x, z)$$
$$f_z = \sin z + e^y \Leftrightarrow f(x, y, z) = ze^y - \cos z + j(x, y)$$

The above information implies that

$$\boxed{f(x, y, z) = ze^y - \cos z + C}.$$

One can compute the gradient of this function and check that $\nabla f = \mathbf{F}$.

(b) The flow of $\mathbf{F}$ along a path is

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

Using the fundamental theorem of line integrals and the potential function we found in part (a):

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= f(\text{end}) - f(\text{start}) \\ &= f(1, 0, \pi) - f(0, 0, 0) \\ &= (\pi + 1) - (-1) \\ &= \boxed{\pi + 2} \end{aligned}$$

(c) Parametrizing the stright line between the end and start gives $\mathbf{r}(t) = \langle t, 0, \pi t\rangle$ and $d\mathbf{r} = \mathbf{r}'(t)\,dt = \langle 1, 0, \pi\rangle\,dt$. $\mathbf{F}(\mathbf{r}(t)) = \langle 0,\ \pi t e^0,\ \sin(\pi t) + e^0\rangle = \langle 0,\ \pi t,\ \sin(\pi t) + 1\rangle$.

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt \\
&= \int_0^1 (\pi\sin(\pi t) + \pi)\,dt \\
&= -\cos(\pi t) + \pi\Big|_0^1 \\
&= (1+\pi) - (-1+0) \\
&= \boxed{\pi + 2}
\end{aligned}$$

■

CONT'D ON REVERSE

Problem 4 (20 pts)
The following questions are not related:

(a) Suppose

$$\int_{\mathcal{C}} 4y\,dx + 7x\,dy = 13$$

where $\mathcal{C}$ is a simple, smooth curve oriented counter-clockwise in the xy-plane that encloses the region $\mathcal{R}$. Given only this information, is it possible to find the area of $\mathcal{R}$? If so, find it and justify your reasoning. If not, explain what additional information you'd need.

<u>**SOLUTION:**</u>

Yes, it is possible to find the area of $\mathcal{R}$. Considering the vector field $\mathbf{F} = \langle 4y,\, 7x\rangle$ and the fact that $\mathcal{C}$ is a simple closed curve, we may invoke Green's Theorem.

Using Green's Theorem (curl-circulation form), we can view the line integral above as the work/flow/circulation of the vector field $\mathbf{F} = \langle 4y, 7x\rangle$ around the curve:

$$\begin{aligned}
13 &= \int_{\mathcal{C}} 4y\,dx + 7x\,dy \\
&= \int_{\mathcal{C}} \langle 4y,\, 7x\rangle \cdot \langle dx,\, dy\rangle \\
&= \int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} \\
&= \iint_{\mathcal{R}} (\nabla \times \mathbf{F})\cdot \mathbf{k}\,dA \text{ (Green's Thm)} \\
&= \iint_{\mathcal{R}} (7-4)\,dA \\
&= \iint_{\mathcal{R}} 3\,dA
\end{aligned}$$

Hence the area of $\mathcal{R} = \displaystyle\iint_{\mathcal{R}} dA = \frac{13}{3}$.

Alternatively you can do a similar calculation using Green's Theorem (flux-divergence form) and the vector field $\mathbf{G} = \langle 7x, -4y\rangle$, and you will reach the same conclusions. ∎

(b) Let $\iint_{\mathcal{R}} dA$ give the area of a region $\mathcal{R}$ in the first quadrant of the xy-plane. (Note, this region is not related to the region $\mathcal{R}$ in part (a)). You are interested in finding the **volume** V**, generated by revolving** $\mathcal{R}$ **about the x- axis**. If $\iint_{\mathcal{R}} g(x,y)\,dA$ is the integral that calculates the **volume** V, determine the integrand $g(x,y)$.

<u>**SOLUTION:**</u>

<u>**Option 1: Use what we've learned about rotating regions in the rz-plane**</u>.

Given any region $\mathcal{S}$ in the rz-plane, we can write its area in terms of double integral(s):

$$Area = \iint_{\mathcal{S}} dz\,dr$$

And the volume obtained by rotating that region around the z-axis is given by integral(s) of the form:

$$Vol = \int_0^{2\pi} \iint\limits_{\mathcal{S}} r\,dz\,dr\,d\theta = 2\pi \iint\limits_{\mathcal{S}} r\,dz\,dr = \iint\limits_{\mathcal{S}} 2\pi r\,dz\,dr$$

In the given problem, we are using the x-axis as the axis of rotation (so it's behaving like z in the formula above). And we are rotating the y-axis around the x-axis, (so y is behaving like r in the formula above).

Thus given any region $\mathcal{R}$ in the xy-plane, we can write its area in terms of double integral(s):

$$Area = \iint\limits_{\mathcal{R}} dx\,dy$$

And the volume obtained by rotating that region around the x-axis is given by integral(s) of the form:

$$Vol = \int_0^{2\pi} \iint\limits_{\mathcal{R}} y\,dx\,dy\,d\theta = 2\pi \iint\limits_{\mathcal{R}} y\,dx\,dy = \iint\limits_{\mathcal{R}} 2\pi y\,dx\,dy$$

Thus

$$\boxed{g(x,y) = 2\pi y}$$

Option 2: Use the Chop up/Approximate/Sum Method we've used throughout Calc 3.

CHOP UP the region $\mathcal{R}$ into tiny subrectangles of area $\Delta A = \Delta x \Delta y$

APPROXIMATE THE VOLUME of rotating one subrectangle around the x-axis: This will result in a tube with rectangular cross sections. If you lay the tube out flat, it will have length $2\pi y$ (the circumference of a circle with radius equal to y) and the area of its cross sections are ΔA. Thus the volume of the tube will be $2\pi y \Delta A$

If we SUM these volumes over the region $\mathcal{R}$ and

TAKE THE LIMIT as $\Delta A \to 0$ we get:

$$Vol = \iint\limits_{\mathcal{R}} 2\pi y\,dA \implies \boxed{g(x,y) = 2\pi y}$$

Option 3: Reverse-engineer the formulas we used for volumes of revolution in calc 2. .

Calc 2 case 1: Washer Method

If the region $\mathcal{R}$ is bounded by curves of the form $y = h(x)$ and $y = j(x)$ where $h(x) \geq j(x)$, in calc 2 we would use the washer method. Since we are rotating about the x-axis, the formula would be

$$\int \pi(R_{\text{out}}^2 - R_{\text{in}}^2)\,dx = \int \pi(h(x)^2 - j(x)^2)\,dx.$$

Calc 2 case 2: Shell Method

However, if the region $\mathcal{R}$ is bounded by curves of the form $x = j(y)$ and $x = k(y)$ with $j(y) > k(y)$, the washer method may not work. In this case we would use the shell method:

$$\int 2\pi R\,h\,dy = \int 2\pi y(j(y) - k(y))\,dy$$

where R is the radius of a given shell and h is the "height" of the shell, or in this case the horizontal length.

Calc 3 Solution

In calc 3 we know that the region $\mathcal{R}$ might be much more complicated, so neither of the above calc 2 formulas would work in full generality. Instead, we want to understand that these formulas are derived from a calc 3 double integral of the form:

$$\iint\limits_{\mathcal{R}} g(x,y)\, dA.$$

Using the calc 2 cases as insight, as well as having mastered the calculus concepts of summing infinitesimal pieces together, the questions one would ask would be,

(i) *Given* $\iint\limits_{\mathcal{R}} g(x,y)\, dA$*, if we integrate with respect to* y *first, how do we get a single washer?*

(ii) *Given* $\iint\limits_{\mathcal{R}} g(x,y)\, dA$ *if we integrate with respect to* x *first, how do we get a single shell?*

To answer (i) notice that a washer or a disk is created by "adding" together i.e. *integrating together*, the circumference of circles with increasing radii. I.e. take a circle, center on the x-axis, circle parallel to the yz-plane, and increase the radius y. We get $\int 2\pi y\, dy$ to create a single washer, then summing these washers horizontally (in the x-direction) gives the volume

$$\iint\limits_{\mathcal{R}} 2\pi y\, dydx.$$

Let's check to see if this formula answers question (ii). To create a shell, we again take the circumference of a circle (center on the x-axis, circle parallel to the yz-plane), but now keep the radius fixed and *integrate* horizontally, so $\int 2\pi y dx$ gives a single shell. Now summing the shells in the y-direction gives the volume

$$\iint\limits_{\mathcal{R}} 2\pi y\, dxdy.$$

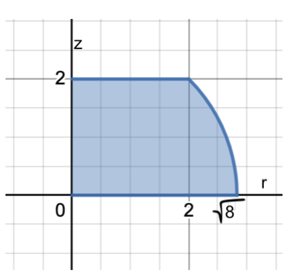
Thus $g(x,y) = 2\pi y$. The astute calc 3 student will check that if the region $\mathcal{R}$ is given by either of the calc 2 cases, then our calc 3 double integral will reduce to the washer method or the shell method after integrating the first variable.

■

Problem 5 (40 pts)
Consider the 3D solid object $\mathcal{E}$ that is bounded on the top by $z = 2$, on the bottom by $z = 0$ and on the sides by $x^2 + y^2 + z^2 = 8$.
Let

$$\mathbf{G} = y\mathbf{i} - x\mathbf{j} + 3z\mathbf{k}$$

(a) Sketch and shade a cross section of the object in the rz-plane. Label axes and any intercepts.
<u>**SOLUTION:**</u>



■

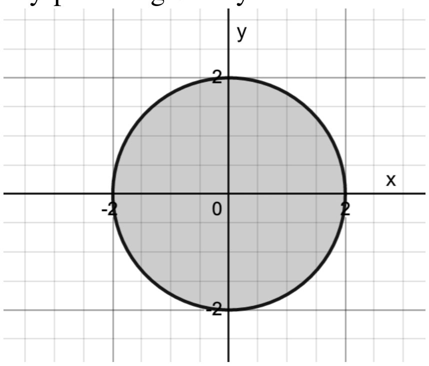
(b) Calculate the volume of the object.
<u>**SOLUTION:**</u>

OPTION 1: Using cylindrical coordinates $dr\, dz\, d\theta$:

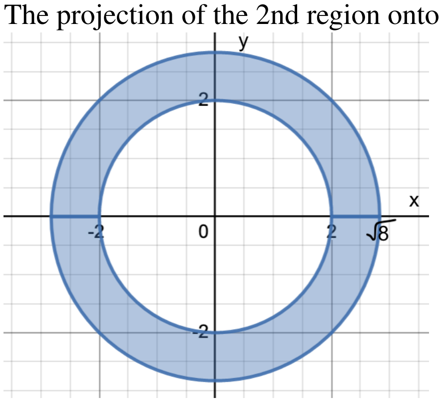
$$V = \int_0^{2\pi}\int_0^{2}\int_0^{\sqrt{8-z^2}} r\,dr\,dz\,d\theta = \boxed{\frac{40}{3}\pi}$$

OPTION 2: Using cylindrical coordinates $dz\,dr\,d\theta$:

Using this ordering we need to break this up into 2 regions. The projection of the first region onto the xy-plane is given by



The projection of the 2nd region onto the xy-plane is given by



$$V = \int_0^{2\pi}\int_0^{2}\int_0^{2} r\,dz\,dr\,d\theta + \int_0^{2\pi}\int_2^{2\sqrt{2}}\int_0^{\sqrt{8-r^2}} r\,dz\,dr\,d\theta = 8\pi + \frac{16}{3}\pi = \boxed{\frac{40}{3}\pi}$$

OR using subtraction

$$V = \frac{1}{2}\left(\frac{4}{3}\pi 8^{3/2}\right) - \int_0^{2\pi}\int_0^{2}\int_2^{\sqrt{8-r^2}} r\,dz\,dr\,d\theta = \frac{32}{3}\pi\sqrt{2} - \frac{8(4\sqrt{2}-5)}{3} = \boxed{\frac{40}{3}\pi}$$

OPTION 3: Using spherical coordinates: $d\rho\,d\phi\,d\theta$:

$$V = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{2\sec\phi} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta + \int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^{\sqrt{8}} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{8}{3}\pi + \frac{32}{3}\pi = \boxed{\frac{40}{3}\pi}$$

OR using subtraction

$$V = \frac{1}{2}\left(\frac{4}{3}\pi 8^{3/2}\right) - \int_0^{2\pi}\int_0^{\pi/4}\int_{2\sec\phi}^{\sqrt{8}} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{32}{3}\pi\sqrt{2} - \frac{8(4\sqrt{2}-5)}{3} = \boxed{\frac{40}{3}\pi}$$

■

(c) Calculate the outward flux of the vector field $\mathbf{G}$ through the entire surface of the object $\mathcal{E}$ using an appropriate Calculus 3 theorem.

<u>**SOLUTION:**</u>

Since the surface $\mathcal{S}$ of $\mathcal{E}$ is closed, and since $\mathbf{G}$ has continuous partial derivatives everywhere, we can use the Divergence Theorem to calculate the net outward flux:

$$\nabla \cdot \mathbf{G} = 3$$

$$Flux = \oiint_{\mathcal{S}} \mathbf{G} \cdot \mathbf{n}\, dS = \iiint_{\mathcal{E}} \nabla \cdot \mathbf{G}\, dV = \iiint_{\mathcal{E}} 3\, dV$$

$$= 3(\text{volume of } \mathcal{E})$$

$$= 3\left(\frac{40\pi}{3}\right) = \boxed{40\pi}$$

■

(d) Verify your answer to part (c) by separately calculating the flux through each part of the bounding surface (i.e. the top, the bottom and the side) and adding them together.

SOLUTION:

$$Flux = \oiint_{\mathcal{S}} \mathbf{G} \cdot \mathbf{n}\, dS = \iint_{\mathcal{S}_{\text{top}}} \mathbf{G} \cdot \mathbf{n}\, dS + \iint_{\mathcal{S}_{\text{bottom}}} \mathbf{G} \cdot \mathbf{n}\, dS + \iint_{\mathcal{S}_{\text{sides}}} \mathbf{G} \cdot \mathbf{n}\, dS$$

Along S_{side}:
Let $g(x, y, z) = x^2 + y^2 + z^2$ and let $\mathbf{p} = \mathbf{k}$
Thus $\nabla g = \langle 2x, 2y, 2z \rangle$, which points in the direction we want since it points outward from the sphere.
Thus

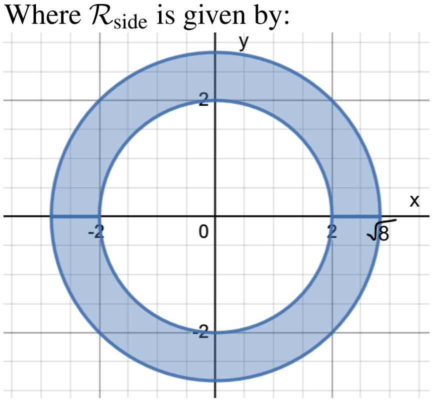
$$\mathbf{n}\, dS = \frac{\nabla g}{|\nabla g \cdot \mathbf{k}|}\, dA = \frac{\langle 2x, 2y, 2z \rangle}{|2z|} = \frac{\langle x, y, z \rangle}{z}\, dA, \text{ since } z > 0 \text{ on the sides}$$

$$\iint_{\mathcal{S}_{\text{side}}} \mathbf{G} \cdot \mathbf{n}\, dS = \iint_{\mathcal{R}_{\text{side}}} \langle y, -x, 3z \rangle \cdot \frac{\langle x, y, z \rangle}{z}\, dA$$

$$= \iint_{\mathcal{R}_{\text{side}}} 3z\, dA$$

$$= \iint_{\mathcal{R}_{\text{side}}} 3\sqrt{8 - x^2 - y^2}\, dA$$

Where $\mathcal{R}_{\text{side}}$ is given by:



Switching to polar

$$= \int_0^{2\pi} \int_2^{2\sqrt{2}} 3\sqrt{8 - r^2}\, r\, dr d\theta$$

$$= 3\,(2\pi) \int_2^{2\sqrt{2}} r\sqrt{8 - r^2}\, dr$$

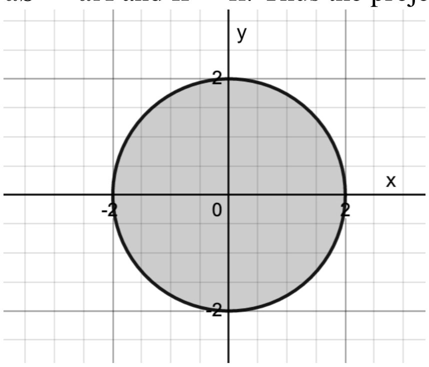
$$= 3\,(2\pi)\left(\frac{8}{3}\right) = 16\pi$$

Along $S_{\textbf{bottom}}$:
$dS = dA$ and $\mathbf{n} = -\mathbf{k}$
Thus

$$\iint\limits_{\mathcal{S}_{\text{top}}} \mathbf{G}\cdot\mathbf{n}\,dS = \iint\limits_{\mathcal{R}_{\text{top}}} \langle y, -x, 3z\rangle \cdot \langle 0, 0, -1\rangle\,dA$$

$$= \iint\limits_{\mathcal{R}_{\text{top}}} 3z\,dA = \iint\limits_{\mathcal{R}_{\text{top}}} 3(0)\,dA = 0 \text{ (since } z = 0 \text{ on the bottom surface.)}$$

Along $S_{\textbf{top}}$:
$dS = dA$ and $\mathbf{n} = \mathbf{k}$. Thus the projection on the xy-plane is given by:



Thus

$$\iint\limits_{\mathcal{S}_{\text{top}}} \mathbf{G}\cdot\mathbf{n}\,dS = \iint\limits_{\mathcal{R}_{\text{top}}} \langle y, -x, 3z\rangle \cdot \langle 0, 0, 1\rangle\,dA$$

$$= \iint\limits_{\mathcal{R}_{\text{top}}} 3z\,dA = \iint\limits_{\mathcal{R}_{\text{top}}} 3(2)\,dA \text{ (since } z = 2 \text{ on the top surface)}$$

$$= 6\iint\limits_{\mathcal{R}_{\text{top}}} dA = 6\left(\text{ Area of } \mathcal{R}_{\text{top}}\right)$$

$$= 6\left(\pi 2^2\right) = 24\pi$$

Thus

$$\text{Total Outward Flux} = \oiint\limits_{\mathcal{S}} \mathbf{G}\cdot\mathbf{n}\,dS = 16\pi + 0 + 24\pi = \boxed{40\pi}$$

■

End Of Exam: Have a great summer!