APPM 2350—Exam 3

Wednesday April 13th, 6:30pm-8pm 2022

This exam has 4 problems. Please start each new problem at the top of a new page in your blue book. Show all your work in your blue book and simplify your answers. Answers with missing or insufficient justification will receive no points. You are allowed one 8.5×11-in page of notes (ONE side). You may NOT use a calculator, smartphone, smartwatch, the Internet or any other electronic device.

Problem 1 (20 points)
Suppose the temperature at any point in the xy-plane is given by

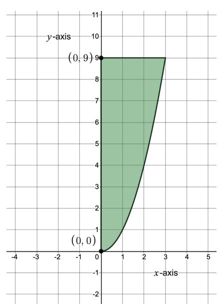
$$T(x,y) = xe^{-y^2}$$

Suppose the area of a region $\mathcal{R}$ is given by

$$\int_0^3 \int_{x^2}^9 dy\,dx$$

(a) Sketch **and shade** the region $\mathcal{R}$ and clearly label your axes and any intercepts.
(b) Find the **average temperature** on the region $\mathcal{R}$. Fully simplify your final answer.

<u>SOLUTION:</u>

(a) 

(b) The average temperature is given by: $T_{ave} = \dfrac{\iint\limits_{\mathcal{R}} T(x,y)\,dA}{\iint\limits_{\mathcal{R}} dA} = \dfrac{\int_0^3 \int_{x^2}^9 xe^{-y^2}\,dy\,dx}{\int_0^3 \int_{x^2}^9 dy\,dx}.$

Considering each integral separately and starting with the denominator:

$$\int_0^3 \int_{x^2}^9 dy\,dx = \int_0^3 \left(9 - x^2\right)\,dx = \left[9x - \frac{x^3}{3}\right]_0^3 = (27-9) - 0 = 18.$$

For the integral in the numerator, it's not possible to integrate this with respect to y (there's no closed form antiderivative for e^{-y^2}).

So instead, we switch the order of integration, rewrite as a $dx\,dy$ integral and calculate:

$$\int_0^3 \int_{x^2}^9 xe^{-y^2}\,dy\,dx = \int_0^9 \int_0^{\sqrt{y}} xe^{-y^2}\,dx\,dy = \int_0^9 \left[\frac{x^2 e^{-y^2}}{2}\right]_0^{\sqrt{y}} dy = \int_0^9 \frac{ye^{-y^2}}{2}\,dy.$$

Letting $u = -y^2$ and $du = -2y\,dy$ then the integral is evaluated as:

$$\int_0^9 \frac{ye^{-y^2}}{2}\,dy = -\frac{1}{2}\int_0^{-81} \frac{e^u}{2}\,du = -\frac{1}{4}\left[e^u\right]_0^{-81}\,du = \frac{1-e^{-81}}{4} = \frac{e^{81}-1}{4e^{81}}.$$

The resulting average temperature is:

$$\frac{\int_0^3 \int_{x^2}^9 xe^{-y^2}\,dy\,dx}{\int_0^3 \int_{x^2}^9 dy\,dx} = \frac{\frac{e^{81}-1}{4e^{81}}}{18} = \boxed{\frac{e^{81}-1}{72e^{81}}}.$$

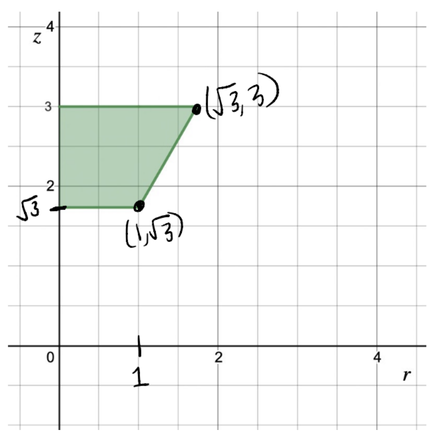
Problem 2 (30 points)
The volume of an object is given by

$$\int_0^{\pi/4}\int_0^1\int_{\sqrt{3}}^3 r\,dz\,dr\,d\theta + \int_0^{\pi/4}\int_1^{\sqrt{3}}\int_{r\sqrt{3}}^3 r\,dz\,dr\,d\theta$$

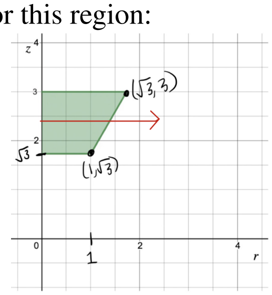
(a) Sketch and shade a 2D cross section of the object in the rz-plane (for any θ such that $0 \le \theta \le \pi/4$). Label the (r, z) coordinates of all corners on the cross section.
(b) Set up, **do not evaluate** equivalent integral(s) to find the volume of the object using:

(i) Cylindrical coordinates in the order $dr\,dz\,d\theta$
(ii) Spherical coordinates in the order $d\rho\,d\phi\,d\theta$

<u>**SOLUTION:**</u>

(a) 

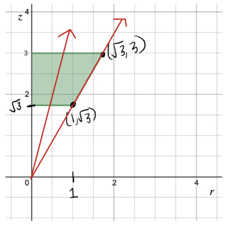
(b) (i) To switch to cylindrical using the ordering $dr\,dz\,d\theta$ we note that
θ bounds remain $0 \le \theta \le \pi/4$
To find the z and r bounds we use the zr-sketch as if we were setting up a double integral for this region:



To find the r bounds we shoot an arrow on our zr-sketch, in the r direction. The arrow enters the region when $r = 0$ and exits when $r = \frac{z}{\sqrt{3}}$
For our z limits we look at the largest and smallest z values on the region, which gives us $\sqrt{3} \le z \le 3$
Thus the integral becomes:

$$\boxed{\int_0^{\pi/4}\int_{\sqrt{3}}^3\int_0^{z/\sqrt{3}} r\,dr\,dz\,d\theta}$$

(ii) To find the ρ limits, shoot an arrow radially outward in the ρ direction on the zr-sketch.



The arrow enters the region when $z = \sqrt{3} \implies \rho\cos\phi = \sqrt{3} \implies \rho = \sqrt{3}\sec\phi$
The arrow exits the region when $z = 3 \implies \rho\cos\phi = 3 \implies \rho = 3\sec\phi$.
To find the ϕ limits, we note that the smallest ϕ on the region occurs when $\phi = 0$.
The largest ϕ occurs when $z = \sqrt{3}r \implies \rho\cos\phi = \sqrt{3}\rho\sin\phi$
$\implies \tan\phi = \frac{1}{\sqrt{3}} \implies \phi = \pi/6$
The θ limits are the same as in cylindrical: $0 \le \theta \le \pi/4$
We replace $r\,dz\,dr\,d\theta$ with $\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$
Thus the integral becomes:

$$\boxed{\int_0^{\pi/4}\int_0^{\pi/6}\int_{\sqrt{3}\sec\phi}^{3\sec\phi} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta}$$

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Problem 3 (30 points)
The following parts are not related:

(a) A thin wire lies along the curve $x = \frac{2}{3}(y-1)^{3/2}$, $1 \le y \le 4$, in the xy-plane. Find the **y-component of the centroid** of the wire. Fully simplify your answer.
<u>**SOLUTION:**</u>

$$\bar{y} = \frac{\int\limits_C y\,ds}{\int\limits_C ds}$$

We start by parameterizing C. Easiest parameterization:
$\mathbf{r}(t) = \langle \frac{2}{3}(t-1)^{3/2}, t\rangle$, $1 \le t \le 4$
$\implies \mathbf{r}'(t) = \langle (t-1)^{1/2}, 1\rangle \implies ||\mathbf{r}'(t)|| = \sqrt{((t-1)^{1/2})^2 + 1} = \sqrt{t}$

$$ds = ||\mathbf{r}'(t)||\,dt = \sqrt{t}\,dt$$

Thus,

$$\bar{y} = \frac{\int\limits_1^4 t\sqrt{t}\,dt}{\int\limits_1^4 \sqrt{t}\,dt} = \frac{\left(\frac{2}{5}t^{5/2}\Big|_1^4\right)}{\left(\frac{2}{3}t^{3/2}\Big|_1^4\right)} = \boxed{\frac{93}{35}}$$

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(b) Let S be any region (e.g. a rectangle, circle, or any other shape) on the plane $3x - 4y + 2z = 6$ that has a surface area equal to 5. Let $\mathcal{R}$ be the projection of the region S onto the **yz-plane**. Find the **area of $\mathcal{R}$**.

$$\text{Surface area} = \iint\limits_S dS = \iint\limits_{\mathcal{R}} \frac{||\nabla g||}{|\nabla g \cdot \hat{\mathbf{p}}|}\,dA$$

$$\implies 5 = \iint\limits_{\mathcal{R}} \frac{||\nabla g||}{|\nabla g \cdot \hat{\mathbf{p}}|}\, dA$$

Since we are projecting onto the yz-plane, $\mathbf{p} = \hat{\mathbf{i}}$
Let $g(x, y, z) = 3x - 4y + 2z$
$\implies \nabla g = \langle 3, -4, 2\rangle \implies ||\nabla g|| = \sqrt{3^2 + (-4)^2 + 2^2} = \sqrt{29}$
and $|\nabla g \cdot \hat{\mathbf{i}}| = 3$
Thus

$$5 = \iint\limits_{\mathcal{R}} \frac{\sqrt{29}}{3}\, dA$$

$$\implies 5 = \frac{\sqrt{29}}{3} \iint\limits_{\mathcal{R}} dA$$

$$\implies \text{Area of } \mathcal{R} = \iint\limits_{\mathcal{R}} dA = \boxed{\frac{15}{\sqrt{29}}}$$

Problem 4 (20 points)
Consider the region, $\mathcal{R}$, that is bounded by

$$y = \sqrt{x}, \quad y = \sqrt{x} + 2, \quad y = 4 - \sqrt{x}, \quad y = 6 - \sqrt{x}$$

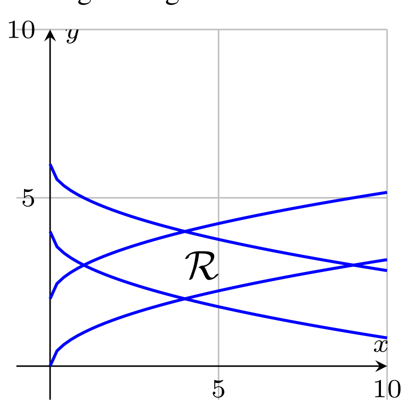
where x and y are measured in meters.
A sprinkler sprays water on the region $\mathcal{R}$ in such a way that the depth of water (in meters) that reaches the point (x, y) in 1 hour is given by

$$g(x, y) = \frac{e^{(y-\sqrt{x})}}{20\sqrt{x}}$$

Use an appropriate uv-transformation to **find the total volume of water** the sprinkler sprays on the region $\mathcal{R}$ in 1 hr. Fully simplify your final answer.
<u>**SOLUTION:**</u>

$$\text{Total Volume} = \iint\limits_{\mathcal{R}} = \frac{e^{(y-\sqrt{x})}}{20\sqrt{x}}\, dA$$

The original region looks like:



One possible transformation based on the boundaries of the region and the integrand: (note there are other possible transformations that will also work)

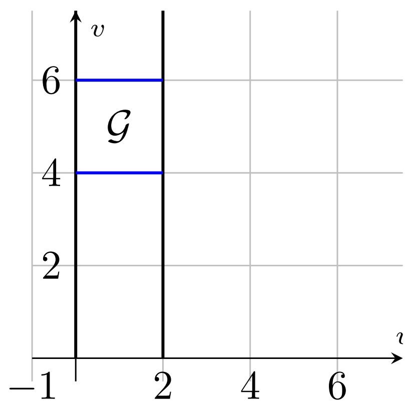
$$u = y - \sqrt{x} \qquad v = y + \sqrt{x}$$

Solving for the xy-transformation yields:

$$x = \left(\frac{v-u}{2}\right)^2 \qquad\qquad y = \frac{u+v}{2}$$

Thus our boundaries convert as follows:

Boundary in xy	Corresponding Boundary in uv
y=$\sqrt{x}$	u=0
y=$\sqrt{x}+2$	u=2
y=4-$\sqrt{x}$	v=4
y=6-$\sqrt{x}$	v=6



Next compute the Jacobian:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{u-v}{2} & \frac{v-u}{2} \\ 1/2 & 1/2 \end{vmatrix} = \frac{u-v}{2}$$

Thus

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\frac{u-v}{2}\right| = \frac{v-u}{2}, \text{ since } u-v = -2\sqrt{x} < 0$$

Making the substitution transforms the integral.

$$\begin{aligned} \int_4^6 \int_0^2 \frac{e^u}{20\frac{v-u}{2}} \frac{v-u}{2}\, dudv &= \frac{1}{20}\int_4^6\int_0^2 e^u\, dudv \\ &= \frac{1}{20}\int_4^6 e^u\Big|_0^2 dv \\ &= \frac{1}{20}\int_4^6 e^2-1\, dv \\ &= \frac{1}{20}(2)(e^2-1) \\ &= \boxed{\frac{e^2-1}{10}m^3} \end{aligned}$$

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End Of Exam