APPM 2350—Exam 2

Wednesday March 9th, 6:30pm-8pm 2022

This exam has 4 problems. Please start each new problem at the top of a new page in your blue book. Show all your work in your blue book and simplify your answers. Answers with missing or insufficient justification will receive no points. You are allowed one 8.5×11-in page of notes (ONE side). You may NOT use a calculator, smartphone, smartwatch, the Internet or any other electronic device.

Problem 1 (30 points)

The temperature at any point in the xy-plane is given by

$$T(x, y) = x^2 + 3y^2 - 2y$$

(a) Find and classify any local max, min or saddle points of $T(x, y)$.

(b) Use Lagrange multipliers to find the location(s) of the hottest and coldest points on the ellipse $x^2 + 2y^2 = 8$.

(c) Given

$$x = 2 + \ln(\theta + 4z), \qquad y = r\cos(\pi r\theta) \qquad z = e^{sr}$$

Use the chain rule to find $\frac{\partial T}{\partial r}$. You can leave the variables x, y and z in your final answer without needing to substitute.

<u>SOLUTION:</u>

(a)

$$\begin{aligned}
\nabla T &= \langle 2x, 6y-2\rangle = \vec{0} \\
\implies 2x &= 0 \qquad\qquad 6y - 2 = 0 \\
x &= 0 \qquad\qquad y = \frac{1}{3}
\end{aligned}$$

$$\text{Critical point: } = \left(0, \frac{1}{3}\right)$$

Classification:

$$\begin{aligned}
D &= \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2 \\
&= \begin{vmatrix} 2 & 0 \\ 0 & 6 \end{vmatrix} \\
&= 12
\end{aligned}$$

Since the $D = 12 > 0$ all critical points are local extrema. Since $f_{xx} > 0$ the graph of f is always concave up. So there is a $\boxed{\text{local min at } \left(0, \frac{1}{3}\right)}$

(b) We can realize the ellipse $x^2 + 2y^2 = 8$ as a level curve of the function

$$g(x, y) = x^2 + 2y^2.$$

Lagrange multipliers tells us that $\nabla f = \lambda \nabla g$, and along with our constraint $g(x, y) = 8$ we obtain the following system:

$$2x = 2\lambda x \tag{1}$$
$$6y - 2 = 4\lambda y \tag{2}$$
$$x^2 + 2y^2 = 8 \tag{3}$$

By equation (1)

$$x = 0 \quad \text{OR} \quad \lambda = 1.$$

By equation (2)

$$(6 - 4\lambda)y - 2 = 0$$
$$y = \frac{1}{3 - 2\lambda}$$

From our solution of (1), if $x = 0$ Eq. (3) gives

$$y = \pm 2$$

giving us the points

$$\boxed{(0, 2) \quad \text{and} \quad (0, -2)}.$$

From our other solution of (1), if $\lambda = 1$ Eq. (2) and (3) give

$$y = 1$$
$$x^2 + 2(1)^2 = 8$$
$$x = \pm\sqrt{6}.$$

Which gives us the points

$$\boxed{(\sqrt{6}, 1) \quad \text{and} \quad (-\sqrt{6}, 1)}.$$

Checking these points gives the following values:

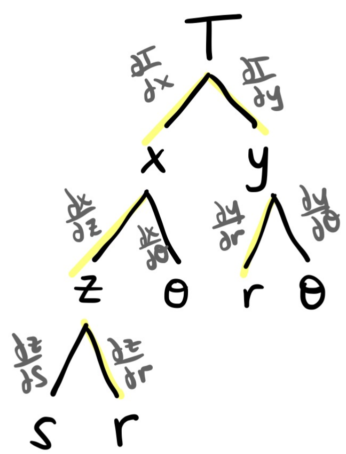
$$\boxed{T(0, 2) = 8, \quad T(0, -2) = 16, \quad T(\sqrt{6}, 1) = 7, \quad T(-\sqrt{6}, 1) = 7}$$

Thus, T has an $\boxed{\text{absolute max of 16 at (0,-2) and an absolute min of 7 at both } (\sqrt{6}, 1) \text{ and } (-\sqrt{6}, 1) \text{ on the ellipse.}}$

(c) Given $T(x, y)$ and the dependencies

$$x = 2 + \ln(\theta + 4z), \qquad y = r\cos(\pi r\theta) \qquad z = e^{sr}$$

we have the following variable dependency tree:



The chain rule states:

$$\frac{\partial T}{\partial r} = \frac{\partial T}{\partial x}\frac{\partial x}{\partial z}\frac{\partial z}{\partial r} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial r}$$

We compute the partial derivatives:

$$\frac{\partial x}{\partial z} = \frac{4}{\theta + 4z}$$
$$\frac{\partial z}{\partial r} = se^{sr}$$
$$\frac{\partial y}{\partial r} = \cos(\pi r\theta) - \pi\theta r\sin(\pi r\theta)$$

Thus

$$\boxed{\frac{\partial T}{\partial r} = (2x)\left(\frac{4se^{rs}}{\theta+4z}\right) + (6y-2)(\cos(\pi r\theta) - \pi\theta r\sin(\pi r\theta))}$$

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Problem 2 (20 points)

The following questions are not related:

(a) Let

$$h(x,y) = \frac{2x^2 - 2y^2}{x+y}$$

(i) The $\lim_{(x,y)\to(1,-1)} h(x,y)$ exists. Determine its value.

(ii) Is $h(x,y)$ continuous at $(1,-1)$? Justify your answer.

<u>**SOLUTION:**</u>

(i) Option 1: Factor and simplify

$$\lim_{(x,y)\to(1,-1)} \frac{2x^2-2y^2}{x+y} \tag{4}$$
$$= \lim_{(x,y)\to(1,-1)} \frac{2\left(x^2-y^2\right)}{x+y} \tag{5}$$
$$= \lim_{(x,y)\to(1,-1)} \frac{2(x-y)(x+y)}{x+y} \tag{6}$$
$$= \lim_{(x,y)\to(1,-1)} 2(x-y) = 2(2) = \boxed{4} \tag{7}$$
$$\tag{8}$$

Option 2:

Since we know the limit exists, we can choose any path that intersects $(1,-1)$ to evaluate the limit.

For example, along the path $x = 1$:

$$\lim_{y\to -1} \frac{2-2y^2}{1+y} = \lim_{y\to -1} \frac{2(1-y)(1+y)}{1+y} = \lim_{y\to -1} 2(1-y) = 2(2) = 4$$

(ii) In order for $h(x,y)$ to be continuous at $(1,-1)$ then $\lim_{(x,y)\to(1,-1)} h(x,y) = h(1,-1)$ must be true. But, while the limit exists,

$\boxed{\text{the function value h(1,-1) does not exist so h(x,y) is not continuous at (1,-1).}}$

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(b) Given

$$f(x,y,z) = \frac{1}{2}e^{4x^2-y^2+z^2}$$

(i) Give the equation of the level surface of $f(x, y, z)$ through the point $(1, 6, 4)$. Then classify (i.e. *give the official name of*) this surface.

(ii) Sketch the level surface you found in part (b)(i). Label your axes and label any intercepts on your sketches.

<u>**SOLUTION:**</u>

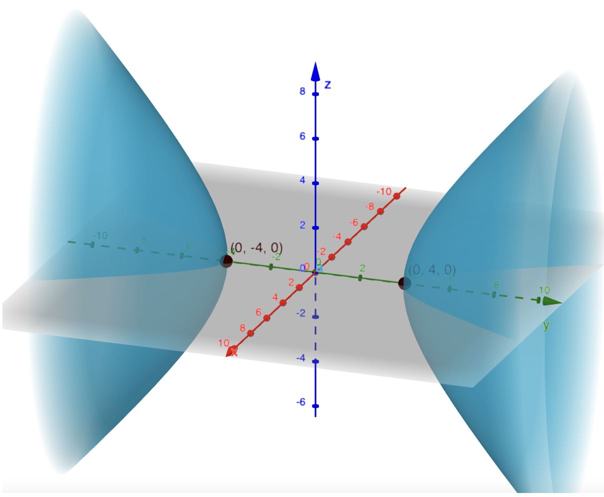
(a) $f(1, 6, 4) = \frac{1}{2}e^{4(1)^2-6^2+4^2} = \frac{1}{2}e^{-16}$

So the level surface is given by $\frac{1}{2}e^{-16} = \frac{1}{2}e^{4x^2-y^2+z^2}$

To classify the level surface we multiply both sides by 2 and take the natural logarithm of both sides to get: $-16 = 4x^2 - y^2 + z^2$. In standard form, $-1 = \frac{x^2}{2^2} - \frac{y^2}{4^2} + \frac{z^2}{4^2}$, this is the equation of a hyperboloid of two sheets.

(b) Sketch the level surface you found in part (b)(i). Label your axes and label any intercepts on your sketches.

This hyperboloid of 2 sheets opens along the y-axis and has y-intercepts $(0, 4, 0)$ and $(0, -4, 0)$.



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Problem 3 (28 points)
Suppose $g(x, y, z)$ is a continuous function with continuous partial derivatives for which

$$g(2, 1, -3) = 6 \text{ and } \nabla g(2, 1, -3) = \langle 5, -2, 10\rangle$$

(a) Find the rate of change of g at the point $(x, y, z) = (2, 1, -3)$ in the direction toward the point $(x, y, z) = (5, -1, -9)$.

(b) Use linearization to approximate $g(2.1, 0.9, -2.8)$.

(c) Let S be the surface $g(x, y, z) = 6$. Find the equation of the plane tangent to S at the point $(x, y, z) = (2, 1, -3)$.

(d) Suppose the surface S in part (c) is the graph of a function, $z = f(x, y)$. Use linearization to approximate $f(2.1, 0.9)$.

<u>**SOLUTION:**</u>

(a) $D_{\hat{\mathbf{u}}}g = \nabla g \cdot \hat{\mathbf{u}}$ In this case, $\hat{\mathbf{u}} = \frac{\langle 3,-2,-6\rangle}{7}$

Thus $D_{\hat{\mathbf{u}}}g(2, 1, -3) = \langle 5, -2, 10\rangle \cdot \frac{\langle 3,-2,-6\rangle}{7} = \boxed{-\frac{41}{7}}$

(b)

$$L(x,y,z) = g(2,1,-3) + g_x(2,1,-3)(x-2) + g_y(2,1,-3)(y-1) + g_z(2,1,-3)(z+3)$$

$$\implies L(x,y,z) = 6 + 5(x-2) - 2(y-1) + 10(z+3)$$

Thus

$$L(2.1, 0.9, -2.8) = 6 + 5(\frac{1}{10}) - 2(-\frac{1}{10}) + 10(\frac{1}{5}) = \boxed{8.7}$$

(c) For a plane we need a point and a normal vector. The point is $(2,1,-3)$. Since S is one level surface of g we know ∇g is normal to S. Thus $\nabla g(2,1,-3) = \langle 5,-2,10\rangle$ a vector normal to the level surface at the point $(2,1,-3)$.

Thus the tangent plane is

$$\boxed{5(x-2) - 2(y-1) + 10(z+3) = 0} \implies z = -3 - \frac{1}{2}(x-2) + \frac{1}{5}(y-1)$$

(d) Method 1:

Since the surface $g(x,y,z) = 6$ is equivalent to the graph of $z = f(x,y)$, the tangent plane we found in part (c) is equivalent to the linearization of $z = f(x,y)$.

The tangent plane from part (c) is:

$$z = -3 - \frac{1}{2}(x-2) + \frac{1}{5}(y-1)$$

$$\implies L(x,y) = -3 - \frac{1}{2}(x-2) + \frac{1}{5}(y-1)$$

Thus

$$L(2.1, 0.9) = -3 - \frac{1}{2}\frac{1}{10} + \frac{1}{5}\frac{-1}{10} = \boxed{-3.07}$$

Method 2:

$L(x,y) = f(2,1) + f_x(2,1)(x-1) + f_y(2,1)(y-1)$.

Since the point $(x,y,z) = (2,1,-3)$ is on the surface, this implies $f(2,1) = -3$.

We now need to find $f_x(2,1)$ and $f_y(2,1)$

Since $z = f(x,y)$ is the graph of $g(x,y,z) = 6$, this implies z can be written as a function of x and y, i.e. $g(x,y,z(x,y)) = 6$.

Taking the partial derivatives of both sides with respect to x we get

$$\frac{\partial}{\partial x} g(x,y,(z(x,y)) = \frac{\partial}{\partial x}(6)$$
$$\implies \frac{\partial g}{\partial x} + \frac{\partial g}{\partial z}\frac{\partial z}{\partial x} = 0 \text{ using the chain rule}$$
$$\implies \frac{\partial z}{\partial x} = -\frac{\frac{\partial g}{\partial x}}{\frac{\partial g}{\partial z}}$$

Since $z = f(x,y)$,

$$\implies \frac{\partial f}{\partial x} = -\frac{\frac{\partial g}{\partial x}}{\frac{\partial g}{\partial z}}$$

$\nabla g(2,1,-3) = \langle 5,-2,10\rangle \implies \frac{\partial g}{\partial x}\Big|_{(2,1,3)} = 5$ and $\frac{\partial g}{\partial x}\Big|_{(2,1,3)} = 10$

Thus

$$f_x(2,1) = -\frac{5}{10} = -\frac{1}{2}$$

Using a similar method we can show

$$\frac{\partial f}{\partial y} = -\frac{\frac{\partial g}{\partial y}}{\frac{\partial g}{\partial z}}$$

Thus

$$f_y(2,1) = -\frac{-2}{10} = \frac{1}{5}$$

Thus

$$L(x,y) = -3 - \frac{1}{2}(x-2) + \frac{1}{5}(y-1)$$

$$\implies L(2.1, 0.9) = -3 - \frac{1}{2}\frac{1}{10} + \frac{1}{5}\frac{-1}{10} = \boxed{-3.07}$$

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Problem 4 (22 points)

Let $f(x,y)$ be a continuous function with continuous partial derivatives.

Suppose the quadratic Taylor approximation of $f(x,y)$ centered at the point $(x,y) = (1,0)$ is given by

$$Q(x,y) = \frac{1}{4}\pi - \frac{3}{4} + x + 2y - \frac{x^2}{4} - xy - y^2$$

For each part below, determine if you have enough information to determine the *exact quantities* described (not estimates) using only $Q(x,y)$ without any other information about $f(x,y)$. If so, find the quantity described. If not, explain what other information you would need.

(a) $f(2,2)$

(b) $f_{xy}(1,0)$

(c) The **value** of the smallest (i.e. most negative) derivative of $f(x,y)$ at the point $(1,0)$ (out of all possible directions)

(d) $f_{xxx}(1,0)$

<u>**SOLUTION:**</u>

(a) Since this quadratic approximation is centered at $(1,0)$, we can only determine exact information about $f(x,y)$ at the point $(1,0)$. Thus it is not possible to determine the exact value of $f(2,2)$ from this info alone. We either need to be given $f(x,y)$ exactly or a Taylor approximation of $f(x,y)$ at the point $(2,2)$ in order to find the exact value of $f(2,2)$.

(b) Since this quadratic approximation is centered at $(1,0)$, we know all first and second order partial derivatives of $f(x,y)$ at the point $(1,0)$ are equal to the first and second order partial derivatives of $Q(x,y)$ at the point $(1,0)$.

$$Q_x(x,y) = 1 - \frac{x}{2} - y$$

$$Q_{xy}(x,y) = -1 \implies Q_{xy}(1,0) = -1$$

$$f_{xy}(1,0) = Q_{xy}(1,0) \implies \boxed{f_{xy}(1,0) = -1}$$

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(c) The value of the smallest derivative of $f(x,y)$ at the point $(1,0)$ out of all possible directions is equivalent to the smallest directional derivative of $f(x,y)$ at the point $(1,0)$.

$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = ||\nabla f||||\mathbf{u}|| \cos\theta = ||\nabla f|| \cos\theta$. This is smallest when $\theta = \pi$. i.e. the value of the smallest derivative of $f(x,y)$ at the point $(1,0)$ is

$$-||\nabla f(1,0)||$$

.

Since this quadratic approximation is centered at $(1,0)$, we know all first and second order partial derivatives of $f(x,y)$ at the point $(1,0)$ match the first and second order partial derivatives of $Q(x,y)$ at the point $(1,0)$.

Thus

$$f_x(1,0) = Q_x(1,0) \text{ and } f_y(1,0) = Q_y(1,0)$$

$$Q_x(x,y) = 1 - \frac{x}{2} - y \implies Q_x(1,0) = \frac{1}{2}$$

$$Q_y(x,y) = 2 - x - 2y \implies Q_y(1,0) = 1$$

Thus

$$-||\nabla Q(1,0)|| = -||\nabla f(1,0)|| = -\sqrt{1/4+1} = \boxed{-\frac{\sqrt{5}}{2}}$$

(d) Since this is a 2nd order quadratic approximation it is not possible to determine information about 3rd order (or any higher than 2nd order) partial derivatives of $f(x,y)$ at the point $(1,0)$ using only $Q(x,y)$. We would either need to know $f(x,y)$ directly, or be given a Taylor approximation of at least 3rd order or higher at the point $(1,0)$.

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End Of Exam
