APPM 2350—Exam 1

Wednesday Feb 9th, 6:30pm-8pm 2022

This exam has 5 problems. Please start each new problem at the top of a new page in your blue book. Show all your work in your blue book and simplify your answers. Answers with no justification will receive no points. You are allowed one 8.5×11-in page of notes (ONE side). You may NOT use a calculator, smartphone, smartwatch, the Internet or any other electronic device.

Problem 1 (26 points)

(a) An engineer measures three points $A = (1, 0, 3)$, $B = (3, 0, 4)$, and $C = (2, 1, 4)$ on a plane P. Find the equation of the plane P. Give your answer in standard (linear) form. (Before moving on, check your final answer by plugging in all 3 points and verifying they are on the plane you found).

(b) You decide to drill along a straight line starting at $Q = (4, 3, 0)$ and perpendicular to the plane P of part (a). Find the point (x, y, z) where the drill intersects the plane P.

(c) Let L be the line you drilled along in part (b), and let L' be the line through the points A and B of part (a). Are L and L' parallel, intersecting, or skew? Justify your answer.

SOLUTION:

(a) To find the equation of a plane, we need a point on the plane and a vector normal to the plane. If we have two vectors that lie in the plane, we can take the cross product to find a normal vector. Let

$$\vec{v}_1 =< 3 - 1, 0 - 0, 4 - 3 >=< 2, 0, 1 >$$

be the vector from A to B that lies in the plane and let

$$\vec{v}_2 =< 2 - 1, 1 - 0, 4 - 3 >=< 1, 1, 1 >$$

be the vector from A to C that lies in the plane. Taking the cross product we get:

$$\vec{n} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = [(0)(1) - (1)(1)]\,\hat{\imath} - [(2)(1) - (1)(1)]\,\hat{\jmath} + [(2)(1) - (0)(1)]\,\hat{k} =$$
$$-\hat{\imath} - \hat{\jmath} + 2\hat{k}.$$

Using any of the given points we get an equation for the plane (in this case using point A):

$$-(x - 1) - y + 2(x - 3) = 0$$

or

$$\boxed{x + y - 2z = -5}$$

(b) The line the drill travels along is in the direction of the normal vector found in part (a) and through Q. The equation of the line is given by:

$$< 4 - t, 3 - t, 2t > .$$

To find the point at which the line intersects the plane we can plug each component of the vector into the equation of the plane and solve for t. We get

$$-(4 - t) - (3 - t) + 2(2t) = 5 \quad (1)$$
$$-4 + t - 3 + t + 4t = 5 \quad (2)$$
$$-7 + 6t = 5 \quad (3)$$
$$6t = 12 \quad (4)$$
$$t = 2 \quad (5)$$

Plugging $t = 2$ into the equation of the line we get:

$$(4-2, 3-2, 2(2)) = \boxed{(2, 1, 4)}$$

(c) From parts (a) and (b): $L =< 4-t, 3-t, 2t >$ and $L' =< 1+2t, 0, 3+t >$. We will begin below by checking if the lines are parallel followed by checking if they intersect. If the lines are not parallel and do not intersect then they are skew.

Check to see if lines are parallel: First we note that the two vectors $< -1, -1, 2 >$ and $< 2, 0, 1 >$ are not scalar multiples of each other since the ratio of the respective x-components is $\dfrac{2}{-1} = -2$ but the ratio of the y-components is $\dfrac{0}{-1} = 0$. So L and L' are not parallel.

Check to see if lines intersect:

Suppose the two lines intersect at point $(x_\circ, y_\circ, z_\circ)$. Let t_1 be the parameter value such that $L(t_1) =< 4-t_1, 3-t_1, 2t_1 >=< x_\circ, y_\circ, z_\circ >$. Let t_2 be parameter value such that $L'(t_2) =< 1+2t_2, 0, 3+t_2 >=< x_\circ, y_\circ, z_\circ >$. Equating each component we get a system of three equations and two unknowns.

$$4 - t_1 = 1 + 2t_2 \tag{6}$$
$$3 - t_1 = 0 \tag{7}$$
$$2t_1 = 3 + t_2 \tag{8}$$

$t_1 = 3$ solves the second equation. Plugging $t_1 = 3$ into equation 3 we get $t_2 = 3$. We check to see if these two values solve the first equation and find that they do not. So the lines do not intersect.

Conclusion:

Since the lines are not parallel and do not intersect then $\boxed{\text{they are skew}}$.

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Problem 2 (16 points) The following questions are not related:

(a) Suppose your roommate uses the force vector $-3\mathbf{i} + 5\mathbf{k}$ to push an object along a straight line path from $(x, y, z) = (3, -1, -7)$ to $(x, y, z) = (1, 5, -4)$. You use the force vector $\mathbf{F}$ to push an object along the same path, but in the optimal direction (i.e. parallel to the path) and you do the same work as your roommate. Find $\mathbf{F}$.

(b) Suppose the curve $\mathcal{C}$ is parameterized with respect to arclength by $\mathbf{r}(s)$. Given only this information, can you determine the distance along the curve $\mathcal{C}$ between $\mathbf{r}(3)$ and $\mathbf{r}(10)$? If so, give the distance (as a real number) and justify your answer. If you do not have enough information, explain what additional information you would need to complete this calculation.

<u>SOLUTION:</u>

(a) Let the force used by your roommate be denoted by

$$\mathbf{G} = \langle -3, 0, 5 \rangle$$

The displacement vector from $(3, -1, -7)$ to $(1, 5, -4)$ is given by

$$\mathbf{D} = \langle -2, 6, 3 \rangle$$

Thus, the total work done by your roommate is

$$W = \mathbf{G} \cdot \mathbf{D} = \langle -3, 0, 5 \rangle \cdot \langle -2, 6, 3 \rangle = 6 + 0 + 15 = 21$$

Option 1:
Since you are applying the force in the direction parallel to the displacement, $\mathbf{F}$ must satisfy

$$\mathbf{F} = c\frac{\langle -2, 6, 3\rangle}{7}$$

To do the same amount of work as your roommate we have:

$$\mathbf{F}\cdot\mathbf{D} = 21 \implies c\frac{\langle -2, 6, 3\rangle}{7}\cdot\langle -2, 6, 3\rangle = 21 \implies 7c = 21 \implies c = 3$$

Thus

$$\mathbf{F} = 3\frac{\langle -2, 6, 3\rangle}{7} = \boxed{\langle -\frac{6}{7}, \frac{18}{7}, \frac{9}{7}\rangle}$$

Option 2: Using a projection
To do the same amount of work with the same displacement but using a force applied in the direction parallel to the displacement, we can project $\mathbf{G}$ onto $\mathbf{D}$ to find what the force $\mathbf{F}$ must be:

$$\mathbf{F} = proj_{\mathbf{D}}\mathbf{G} = \frac{\mathbf{D}\cdot\mathbf{G}}{||\mathbf{D}||}\frac{\mathbf{D}}{||\mathbf{D}||} = \boxed{\frac{21}{7}\frac{\langle -2, 6, 3\rangle}{7} = \langle -\frac{6}{7}, \frac{18}{7}, \frac{9}{7}\rangle}$$

You can check that this force does in fact result in the same work:

$$W = \mathbf{F}\cdot\mathbf{D} = \langle -\frac{6}{7}, \frac{18}{7}, \frac{9}{7}\rangle\cdot\langle -2, 6, 3\rangle = \frac{12 + 108 + 27}{7} = 21$$

(b) Since $\mathbf{r}(s)$ is an arclength parameterization, we know $||r'(s)|| = 1$ for all s.
Thus,

$$arclength = \int_3^{10} ||r'(s)||\, ds = \int_3^{10} 1\, ds = 10 - 3 = \boxed{7}$$

Another way to explain this: At the position $\mathbf{r}(s)$, s measures the distance traveled along the curve (starting from $s = 0$). So the total distance traveled along the curve between $s = 10$ and $s = 3$ is $10 - 3 = 7$.

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Problem 3 (26 points)
A new drone designed for planet exploration is designed with a dual propulsion system with both fixed wing propellers and a rear thruster. Navigational systems on the drone track the direction of travel as

$$\hat{\mathbf{T}}(t) = \left\langle \frac{1}{1+2t^2}, \frac{2t}{1+2t^2}, 1 - \frac{1}{1+2t^2}\right\rangle$$

where t is given in seconds and the speed of the drone at time t is give by $f(t) = 1 + 2t^2 \quad \frac{\text{miles}}{\text{sec}}$

(a) Find the distance the drone travels along its path from $t = 1$ to $t = 2$.
(b) If the drone's position when $t = 1$ is $\langle 2, 1, -\frac{1}{3}\rangle$ find the position of the drone when $t = 3$.
(c) Find the (x, y, z) coordinate(s) of the drone's location(s) when $a_T = 8$.

SOLUTION:

(a) From the problem context $f(t) = ||\mathbf{v}(t)|| = 1 + 2t^2$. Using the formula for arclength we compute the distance as

$$\begin{aligned} s = \int_1^2 ||\mathbf{v}(t)||dt &= \int_1^2 (1+2t^2)dt \\ &= t + \frac{2}{3}t^3 \Big|_1^2 \\ &= \left(2 + \frac{16}{3}\right) - \left(1 + \frac{2}{3}\right) \\ &= \frac{17}{3} \text{ miles} \end{aligned}$$

(b)

$$\begin{aligned} \mathbf{v}(t) &= ||\mathbf{v}(t)||\hat{\mathbf{T}}(t) = \langle 1, 2t, 2t^2 \rangle \\ \mathbf{r}(t) &= \int \mathbf{v}(t)dt = \left\langle t, t^2, \frac{2}{3}t^3 \right\rangle + \vec{c} \\ \mathbf{r}(1) &= \left\langle 1, 1, \frac{2}{3} \right\rangle + \vec{c} = \left\langle 2, 1, -\frac{1}{3} \right\rangle \\ \vec{c} &= \langle 1, 0, -1 \rangle \\ \mathbf{r}(t) &= \left\langle t+1, t^2, \frac{2}{3}t^3 - 1 \right\rangle \\ \mathbf{r}(3) &= \langle 4, 9, 17 \rangle \end{aligned}$$

(c) $a_T = \frac{d}{dt}||\mathbf{v}(t)|| = 4t$. $a_T = 8$ when $t = 2$. $\mathbf{r}(2) = \left\langle 3, 4, \frac{13}{3} \right\rangle$.

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Problem 4 (16 points)
You are managing a spacecraft that has been launched into space to fly to the space station. You have determined that the space station landing spot has the coordinates $(x, y, z) = (8, -2, 0)$. The spacecraft is currently at the point $(x, y, z) = (4, -6, 3)$ and traveling along a straight line in the direction of $2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

(a) How far is the spacecraft currently from the landing spot?

(b) Will the spacecraft hit the landing spot if it continues along its current trajectory? Explain why or why not. If not, find the (x, y, z) coordinates of the point on the spacecraft's path that is closest to the landing spot.

SOLUTION:

(a) The distance between $(x, y, z) = (8, -3, 0)$ and $(4, -6, 3)$ is given by

$$D = \sqrt{(8-4)^2 + (-2+6)^2 + (0-3)^2} = \boxed{\sqrt{41}}$$

(b) To determine if the spacecraft will hit the landing spot, we find the spacecraft's path, which is a line through the point $(4, -6, 3)$ in the direction $\langle 2, 1, -2 \rangle$.

Thus, the spacecraft's position is given by
$\mathbf{r}(t) = \langle 4 + 2t, -6 + t, 3 - 2t \rangle$.
To determine if this ever intersects the point $(8, -2, 0)$ we see if there is a solution to:
$4 + 2t = 8$
$-6 + t = -2$
$3 - 2t = 0$
The first equation implies $t = 2$. But $t = 2$ is not a solution to the 2nd and third equations.
Therefore, the $\boxed{\text{spacecraft will NOT hit the landing spot}}$.

Determining point on path that is closest to landing spot:

Option 1: Using Projections

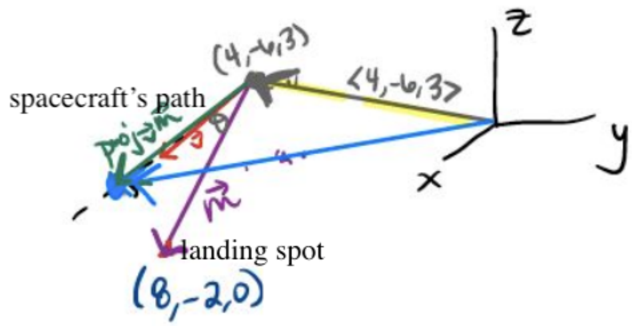
Let

$\mathbf{v} = \langle 2, 1, -2 \rangle$ and

$\mathbf{m} = \langle 8-4, -2+6, 0-3 \rangle = \langle 4, 4, -3 \rangle$

(where $\mathbf{m}$ is a vector from the spacecraft's current position to its landing spot).

Thus $proj_{\mathbf{v}}\mathbf{m} = \frac{\mathbf{v}\cdot\mathbf{m}}{||\mathbf{v}||^2}\mathbf{v} = \frac{18}{9}\langle 2, 1, -2 \rangle = \langle 4, 2, -4 \rangle$



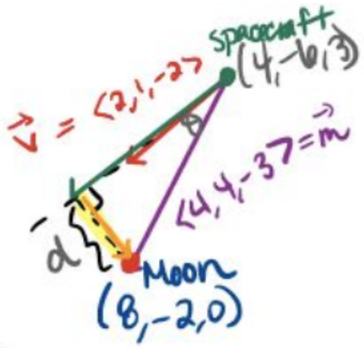
To find the location of the closest point, we can use vector addition:

$\underbrace{\langle 4, -6, 3 \rangle}_{\text{position vector of current location}} + proj_{\vec{v}}\vec{m} =$ position vector of closest point on the line

ie $\langle 4, -6, 3 \rangle + \langle 4, 2, -4 \rangle = \langle 8, -4, -1 \rangle$

Thus $\boxed{(x, y, z) = (8, -4, -1)}$ is the coordinates of the pt on the spacecraft's path that is closest

Option 2: Minimize the distance



minimize the distance between $(8,-2,0)$ and
$(4+2t, -6+t, 3-2t)$

$$D(t) = \sqrt{(4+2t-8)^2 + (-6+t+2)^2 + (3-2t+0)^2}$$
$$= \sqrt{(-4+2t)^2 + (-4+t)^2 + (3-2t)^2}$$
$$= \sqrt{16-16t+4t^2+16-8t+t^2+9-12t+4t^2}$$
$$= \sqrt{9t^2-36t+41}$$

To find the minimum we can solve where $D'(t) = 0$ and then use the 2nd derivative test to justify it's a minimum. However, the minimum of this function will occur at the same t-value as the minimum of $(distance)^2$. Thus, it's algebraically more efficient to minimize $(distance)^2$.

Let $D_{sq} = 9t^2 - 36t + 41$.

This is a parabola that opens up. We can either find the vertex (which will be its min), or solve

$D'_{sq}(t) = 0 \implies 18t - 36 = 0 \implies t = 2.$.

When $t = 2$, $\mathbf{r}(2) = \langle 8, -4, -1 \rangle$

Thus the closest point is $\boxed{\text{(8, -4, -1)}}$

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Problem 5 (16 points)

The following questions are not related:

(a) Give an example of ONE vector-valued function $\mathbf{r}(t)$ that traces out the curve of intersection of the surfaces $x = 4y^2$ and $x^2 = 2z - 6y^2$

(b) Give an example of ONE vector-valued function, $\mathbf{r}(t)$, $t \geq 0$ with ALL of these specified properties (or explain why such a function does not exist):

- $\mathbf{r}(0) = \langle 0, 4, 0 \rangle$
- and $\kappa(t) = \frac{1}{4}$ for all $t \geq 0$
- and $\mathbf{B}(t) = \mathbf{i}$ for all $t \geq 0$
- and $a_T(t) \neq 0$ for $t > 0$

<u>**SOLUTION:**</u>

(a) There are an infinite number of possible parameterizations. To check if a parameterization is correct, plug it back into both surfaces and make sure it works in both.

$$\boxed{\text{One possibility: } \mathbf{r}(t) = \langle 4t^2, t, 8t^4 + 3t^2 \rangle}$$

(b)

- The condition $\kappa(t) = \frac{1}{4}$ for all t tells us this is a circle of radius 4. (Note that using only this condition it could also possibly be a helix, but the condition $\mathbf{B}(t) = \mathbf{i}$ for all t tells us this path lies in a plane perpendicular to the x-axis, thus eliminating the helix option).
- The condition $\mathbf{r}(0) = \langle 0, 4, 0 \rangle$ tells us to start our circle at the point $(0, 4, 0)$ when $t = 0$.
- Since the parameterization starts at $(0, 4, 0)$, the condition $\mathbf{B}(t) = \mathbf{i}$ tells us the circle is traced counterclockwise in the yz-plane (when viewed from the positive x axis), and the circle must be centered at the origin.
- The last condition tells us $a_T(t) \neq 0$ i.e. $\frac{d}{dt}||v(t)|| \neq 0$ for $t > 0$, i.e. that the speed of the parameterization cannot be constant.

Thus we want to parameterize a circle with radius 4 in the yz-plane, centered at the origin that starts at $(0, 4, 0)$ and traverses counterclockwise (when viewed from the positive x-axis), and that does not have constant speed.

There are an infinite number of possible parameterizations.

$$\boxed{\text{One possibility: } \mathbf{r}(t) = \langle 0, 4\cos(t^2), 4\sin(t^2) \rangle, \;\; t \geq 0}$$

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End Of Exam