APPM 2350—Final Exam (Cumulative)
Wednesday May 5th, 10:30am-1pm 2021

Show all your work and simplify your answers. Answers with no justification will receive no points.

If you are asked to fully set-up but not evaluate integrals, this means to find the bounds of integration, fully simplify the integrands, and set-up in the coordinate system that leads to the simplest and fewest integrals. For line integrals and surface integrals, this also means correctly converting ds and dS.

Problem 1 (24 pts)

Consider a particle moving along a path $\mathcal{C}$ in space, where distance is measured in meters, time is measured in seconds, and the temperature at any point is given by the function $T(x, y, z)$ (in degrees Fahrenheit).

At a particular instant in time, t^* (and only at that instant in time), the position of the particle is $\mathbf{r}(t^*) = 3\mathbf{i} - 1\mathbf{j} + 2\mathbf{k}\ m$, its velocity is $\mathbf{v}(t^*) = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}\ \frac{m}{s}$ and its acceleration is $\mathbf{a}(t^*) = 2\mathbf{j}\ \frac{m}{s^2}$.

In addition, you are provided the following information about $T(x, y, z)$:
$T(2, 2, 1) = 10$°F $\quad \nabla T(2, 2, 1) = 4\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}\ \frac{°F}{m}$

$T(3, -1, 2) = 6$°F $\quad \nabla T(3, -1, 2) = -1\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}\ \frac{°F}{m}$

If possible, calculate each of the quantities below using only the information provided above. If there is not enough information provided, then write "Not enough info." (Note that you *may not need to use* all the information provided).

(a) At time $t = t^*$, what is the rate of change of the particle's temperature with respect to time t? *Include units.*
(b) If the particle continues along the path $\mathcal{C}$ for $\Delta t = 0.1$ seconds, by approximately how much will the temperature change? *Include units.*
(c) At time $t = t^*$, what is the rate of change of the particle's temperature with respect to distance traveled (arc length s) along its path? *Include units.*
(d) Calculate the curvature of the path at the position described. *Include units.*

SOLUTION:

(a) Using the chain rule,

$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt}$$

At the location when $t = t^*$,

$$\langle \frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z} \rangle = \nabla T(3, -1, 2) = -1\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}\ \ \frac{°F}{m}$$

and

$$\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle = \mathbf{v}(t^*) = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}\ \ \frac{m}{s}$$

.

Thus

$$\boxed{\frac{dT}{dt} = (-1)(2) + (2)(2) + (3)(1) = 5\ \ \frac{°F}{s}}$$

(b)

$$\Delta T \approx \frac{dT}{dt}\Delta t$$
$$\boxed{= (5)(0.1) = 0.5°F}$$

(c) This is asking for $\frac{dT}{ds}$ in the direction of the particle.

Option 1:

$$\frac{dT}{ds} = D_u T = \nabla T(3,-1,2) \cdot \mathbf{u}$$

At time $t = t^*$, the particle is moving in the direction $\mathbf{u} = \frac{\mathbf{v(t^*)}}{||\mathbf{v(t^*)}||} = \frac{2\mathbf{i}+2\mathbf{j}+\mathbf{k}}{\sqrt{2^2+2^2+1^2}} = \frac{2\mathbf{i}+2\mathbf{j}+\mathbf{k}}{3}$

Thus

$$\boxed{D_u T = (-1)(\frac{2}{3}) + (2)(\frac{2}{3}) + (3)(\frac{1}{3}) = \frac{5}{3} \ \frac{{}^\circ F}{m}}$$

Option 2:

$$\boxed{\frac{dT}{ds} = \frac{dT}{dt}\frac{dt}{ds} = \frac{dT}{dt}\frac{1}{||\mathbf{v}(t^*)||} = (5)\frac{1}{3} = \frac{5}{3} \ \frac{{}^\circ F}{m}}$$

(d)

$$\kappa = \frac{||\mathbf{v} \times \mathbf{a}||}{||\mathbf{v}||^3} = \frac{|| < -2, 0, 4 > ||}{3^3} = \frac{2\sqrt{5}}{27} \ \frac{1}{m}$$

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Problem 2 (18 pts)

Water is flowing down a vertical cylindrical pipe of radius 3 inches, where the pipe is represented by the cylinder $x^2 + y^2 = 9$, bounded below by the xy-plane.

(a) Suppose the velocity vector field of the water at the outlet (i.e. the bottom opening) of the pipe is given by $\mathbf{v} = \langle 0, 0, -9 \rangle \ \frac{in}{min}$.
Find the total volume of water that flows out of the outlet of the pipe over 10 minutes. *Include units.*

(b) If the velocity vector field at the outlet of the pipe is instead given by $\mathbf{v} = \langle 0,\ 0,\ x^2 + y^2 - 9 \rangle \ \frac{in}{min}$, find the total volume of water that now flows out of the outlet of the pipe over 10 minutes. *Include units.*
(Fun fact: this velocity field models how water actually behaves when flowing through a pipe).

<u>**SOLUTION:**</u>

To find the total volume, we want (Flux)*(Time)

$$Flux = \iint_S (\mathbf{v} \cdot \mathbf{n})\, dS =$$

where $S = R$ is the circle in the xy-plane of radius 3 and $\mathbf{n} = \langle 0, 0, -1 \rangle$, since we are looking forward downward flux. (Notice that since $S = R, dS = dA$)

(a)

$$Flux = \iint_R 9\, dA = 9(\text{area of R}) = 81\pi \ \frac{in^3}{min}$$

Thus, total volume over 10 min = (flux)(time) = $\boxed{810\pi \ \ in^3}$

(b)

$$Flux = \iint_R \langle 0, 0, x^2 + y^2 - 9 \rangle \cdot \langle 0, 0, -1 \rangle\, dA = \iint_R (9 - x^2 - y^2)\, dA$$

Switch to polar

$$= \int_0^{2\pi} \int_0^3 (9 - r^2) r\, dr\, d\theta = \left(\int_0^{2\pi} d\theta \right) \left(\int_0^3 (9r - r^3)\, dr \right)$$

$$= (2\pi) \left(\frac{9r^2}{2} - \frac{r^4}{4} \Big|_0^3 \right)$$

$$= \frac{81\pi}{2} \frac{in^3}{min}$$

Thus, total volume over 10 min = $\boxed{405\pi \ \ in^3}$

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Problem 3 (15 pts)

A company produces three products. Suppose that the profit of the company, in millions of dollars, is $P(x, y, z) = 4x + 8y + 6z$, where x, y and z are the number of units (in thousands) of the three products sold. Manufacturing constraints force x, y and z to satisfy $x^2 + 4y^2 + 2z^2 \leq 800$. What is the maximum profit the company can earn? Provide full justification and simplify your answer. *Include units*.

SOLUTION:

We have to optimize the function $P(x, y, z)$ *inside* the given ellipsoidal solid *and* on the boundary surface.

Note that $\nabla P(x, y, z) = \langle 4, 8, 6\rangle$, which clearly always exists and never equals the zero vector, so there are no critical points in the interior $x^2 + 4y^2 + 2z^2 < 800$ to worry about.

As for on the ellipsoid boundary surface, let $g(x, y, z) = x^2 + 4y^2 + 2z^2$. Using Lagrange multipliers, we try to solve the system

$$\nabla f = \lambda \nabla g \qquad \text{and} \qquad g(x, y, z) = 800$$

or

$$\langle 4, 8, 6\rangle = \lambda\langle 2x, 8y, 4z\rangle \qquad \text{and} \qquad g(x, y, z) = 800$$

or

$$\begin{aligned} 4 = 2\lambda x &\implies x = \frac{2}{\lambda} \\ 8 = 8\lambda y &\implies y = \frac{1}{\lambda} \\ 6 = 4\lambda z &\implies z = \frac{3}{2\lambda} \\ g(x, y, z) = 800 & \end{aligned}$$

(Note we may divide by λ since $\lambda = 0$ is impossible in this system, for otherwise the first equation would be $4 = 0$, which is false)

Plugging these expressions for x, y and z into the constraint equation yields

$$\begin{aligned} \left(\frac{2}{\lambda}\right)^2 + 4\left(\frac{1}{\lambda}\right)^2 + 2\left(\frac{3}{2\lambda}\right)^2 &= 800 \\ \frac{4}{\lambda^2} + \frac{4}{\lambda^2} + \frac{9}{2\lambda^2} &= 800 \\ \frac{25}{2\lambda^2} &= 800 \qquad\qquad 8 + 8 + 9 = 25 \\ \lambda^2 &= \frac{1}{800} \cdot \frac{25}{2} = \frac{1}{64} \\ |\lambda| &= \frac{1}{8} \\ \lambda &= \pm\frac{1}{8} \end{aligned}$$

But we may ignore $\lambda = -\frac{1}{8}$ in this question because it would result in a negative number of products made.

Thus $\lambda = \frac{1}{8}$, and solving for x, y and z as above gives

$$x = 16 \qquad y = 8 \qquad z = 12$$

and the maximum profit the company can earn is

$$P(16, 8, 12) = 4(16) + 8(8) + 6(12) = \$200 \text{ million}$$

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Problem 4 (38 pts)

Consider the force field

$$\mathbf{F} = xy\mathbf{i} - y\mathbf{j}$$

Suppose a closed, curved wire lies along the part of $x = 3 - y^2$ from $(2, -1)$ to $(-1, 2)$ followed by the line segment between $(-1, 2)$ and $(2, -1)$. The density at any point on the wire is proportional to the square of the distance from the origin at that point. Fully set up (but do not evaluate) integral(s) to find the quantities described below. **Please see the note at the top of the exam for a reminder of what we mean by fully set up**.

(a) Fully set up (do not evaluate) integral(s) to find the mass of the wire. Use k for your proportionality constant.

(b) Fully set up (do not evaluate) integral(s) to find the work done by **F** counterclockwise along the path of the wire via the following methods:

  (i) Setting up line integral(s).

  (ii) Using an appropriate theorem from Calculus 3 (again, just set up, do not actually evaluate).

SOLUTION:

(a)

$$M = \int_{\mathcal{C}} \delta(x, y) ds = \int_{\mathcal{C}_1} k(x^2 + y^2) ds + \int_{\mathcal{C}_2} k(x^2 + y^2) ds$$

where the parabolic curve is given by $\mathcal{C}_1$ and the line segment by $\mathcal{C}_2$.

To set-up these integrals, we need to parameterize the line segments and rewrite everything in terms of one variable.

Possible parameterization of $\mathcal{C}_1$:

$$\mathbf{r_1}(t) = \langle 3 - t^2, t \rangle, -1 \le t \le 2$$

$$\implies \mathbf{v_1}(t) = \langle -2t, 1 \rangle, \text{ and } ||\mathbf{v_1}|| = \sqrt{4t^2 + 1}$$

Possible parameterization(s) of $\mathcal{C}_2$:

$$\mathbf{r_2}(t) = \langle t, 1 - t \rangle, -1 \le t \le 2$$

$$\implies \mathbf{v_2}(t) = \langle 1, -1 \rangle, \text{ and } ||\mathbf{v_2}|| = \sqrt{2}$$

(or another possibility: is $\mathbf{r_2}(t) = \langle 3t - 1, 2 - 3t \rangle, 0 \le t \le 1$)

Changing variables, we have $ds = ||\mathbf{v}||\, dt$

Thus

$$\int_{\mathcal{C}_1} k(x^2 + y^2) ds = k \int_{-1}^{2} \left((3 - t^2)^2 + t^2\right) \sqrt{4t^2 + 1}\, dt = k \int_{-1}^{2} (9 - 5t^2 + t^4) \sqrt{4t^2 + 1}\, dt$$

and

$$\int_{\mathcal{C}_2} k(x^2 + y^2) ds = k \int_{-1}^{2} (t^2 + (1 - t)^2) \sqrt{2}\, dt = \sqrt{2} k \int_{-1}^{2} (2t^2 - 2t + 1)\, dt$$

Thus, one possible set-up is:

$$\boxed{Mass = k\int_{-1}^{2}(9-5t^2+t^4)\sqrt{4t^2+1}\,dt + \sqrt{2}k\int_{-1}^{2}(2t^2-2t+1)\,dt}$$

Another equivalent set-up is:

$$\boxed{Mass = k\int_{-1}^{2}(9-5t^2+t^4)\sqrt{4t^2+1}\,dt + 3\sqrt{2}k\int_{0}^{1}(18t^2-18t+5)\,dt}$$

(b) The force field is given as $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$, where $P(x,y) = xy$ and $Q(x,y) = -y$. The work done by $\mathbf{F}$ counterclockwise along the closed curve $\mathcal{C}$ is $W = \oint_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r}$.

(i) The work done can be expressed as the sum of two line integrals: one corresponding to the parabolic curve $\mathcal{C}_1$ and the other corresponding to the line segment $\mathcal{C}_2$. Recall the common parameterization of $\mathcal{C}_1$ and the two common parameterizations of $\mathcal{C}_2$ from part (a):

$$\begin{aligned}\mathcal{C}_1:\ & x(t) = 3-t^2,\ \ y(t) = t,\ \ -1\le t\le 2\\ & x'(t) = -2t,\ \ y'(t) = 1\end{aligned}$$

$$\begin{aligned}\mathcal{C}_2:\ & x(t) = t,\ \ y(t) = 1-t,\ \ -1\le t\le 2\\ & x'(t) = 1,\ \ y'(t) = -1\end{aligned}$$

$$\begin{aligned}\mathcal{C}_2:\ & x(t) = 3t-1,\ \ y(t) = 2-3t,\ \ 0\le t\le 1\\ & x'(t) = 3,\ \ y'(t) = -3\end{aligned}$$

Those parameterizations lead to the following line integrals:

$$\begin{aligned}\int_{\mathcal{C}_1} Pdx + Qdy &= \int_{-1}^{2}\big[x(t)y(t)x'(t) - y(t)y'(t)\big]\,dt = \int_{-1}^{2}\big[(3-t^2)(t)(-2t) - (t)(1)\big]\,dt\\ &= \int_{-1}^{2}\big[2t^2(t^2-3) - t\big]\,dt = \int_{-1}^{2}\big(2t^4 - 6t^2 - t\big)\,dt\end{aligned}$$

$$\begin{aligned}\int_{\mathcal{C}_2} Pdx + Qdy &= \int_{-1}^{2}\big[x(t)y(t)x'(t) - y(t)y'(t)\big]\,dt = \int_{-1}^{2}\left[(t)(1-t)(1) - (1-t)(-1)\right]dt\\ &= \int_{-1}^{2}\big(t - t^2 + 1 - t\big)\,dt = \int_{-1}^{2}\big(1-t^2\big)\,dt\end{aligned}$$

$$\begin{aligned}\int_{\mathcal{C}_2} Pdx + Qdy &= \int_{0}^{1}\big[x(t)y(t)x'(t) - y(t)y'(t)\big]\,dt = \int_{0}^{1}\left[(3t-1)(2-3t)(3) - (2-3t)(-3)\right]dt\\ &= \int_{0}^{1} 3\big[(-9t^2+9t-2) + (2-3t)\big]\,dt = \int_{0}^{1} 3(-9t^2+6t)dt = \int_{0}^{1} 9t(2-3t)dt\end{aligned}$$

The preceding line integrals can be used to construct the following two expressions for the work done (these are not the only possible solutions, but they are based on the most common parameterizations of $\mathcal{C}_1$ and $\mathcal{C}_2$):

$$W = \oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}_1} Pdx + Qdy + \int_{\mathcal{C}_2} Pdx + Qdy$$

$$= \int_{-1}^{2} \left(2t^4 - 6t^2 - t\right) dt + \int_{-1}^{2} \left(1 - t^2\right) dt = \boxed{\int_{-1}^{2} \left(2t^4 - 7t^2 - t + 1\right) dt}$$

$$W = \oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}_1} Pdx + Qdy + \int_{\mathcal{C}_2} Pdx + Qdy$$

$$= \boxed{\int_{-1}^{2} \left(2t^4 - 6t^2 - t\right) dt + \int_{0}^{1} 9t(2 - 3t)dt}$$

(ii) The Flow Vector form of Green's Theorem could be used to calculate the work done by evaluating the following double integral over the region $\mathcal{D}$ contained in $\mathcal{C}$:

$$W = \oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}[-y] = 0, \quad \frac{\partial P}{\partial y} = \frac{\partial}{\partial y}[xy] = x$$

$$\Rightarrow \quad \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_{\mathcal{D}} -x dA$$

Integrating over the region $\mathcal{D}$ with respect to x would require two double integrals, which is not in fully-simplified form (as defined in the instructions at the top of the exam), while integrating with respect to y leads to an expression involving only one double integral. The equation for the parabolic arc $\mathcal{C}_1$ can be expressed as $x = 3 - y^2$ and the equation for the line segment $\mathcal{C}_2$ can be expressed as $x = 1 - y$. Those expressions lead to the following fully-simplified result:

$$W = \boxed{-\int_{-1}^{2} \int_{1-y}^{3-y^2} x dx dy}$$

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Problem 5 (40 pts)

Let $\mathbf{V}$ be a vector field and let $\mathcal{S}$ be the surface of a sphere with radius 6 centered at the origin.
Helmholtz Theorem says that any vector field on a closed region (like a sphere) can be decomposed into two special component fields, such that
$\mathbf{V} = \mathbf{V_1} + \mathbf{V_2}$, where $\mathbf{V_1} = -\nabla f(x, y, z)$ and $\mathbf{V_2} = \nabla \times \mathbf{G}$.

Suppose $f(x, y, z) = 3z^2 - x^2 - y^2$ and $\mathbf{G} = \langle y^2 + z^2,\, xyz,\, 0\rangle$.

(a) Which vector field(s) ($\mathbf{V_1}$ and/or $\mathbf{V_2}$) contribute to the total flow of $\mathbf{V}$ around a great circle of the sphere? Explain your reasoning. (Reminder: a great circle is a circle formed by intersecting a sphere with a plane directly through the center of the sphere).
(b) Find the flow of $\mathbf{V}$ around the great circle of the sphere in the yz-plane, in a counterclockwise direction when viewed from the positive x-axis.

(c) Which vector field(s) ($\mathbf{V_1}$ and/or $\mathbf{V_2}$) contribute to the total flux of $\mathbf{V}$ through the surface of the sphere? Explain your reasoning.
(d) Find the total outward flux of $\mathbf{V}$ through the surface of the sphere.

SOLUTION:

(a) There are two ways to think about this question.

- Option 1: From the Fundamental Theorem of Line integrals we have

$$\begin{aligned}\oint_{\mathcal{C}} \mathbf{V_1} \cdot \mathbf{d}r &= -\oint_{\mathcal{C}} \nabla f \cdot \mathbf{d}r \\ &= f(\mathcal{C}_{\text{End}}) - f(\mathcal{C}_{\text{Start}}) \\ &= 0\end{aligned}$$

  where we have used the fact that $\mathcal{C}_{\text{End}} = \mathcal{C}_{\text{Start}}$ on a closed curve (like a great circle of the sphere). So, only $\mathbf{V_2}$ can contribute to the flow around a great circle of the sphere.

- Option 2: From Stoke's Theorem we have

$$\oint_{\mathcal{C}} \mathbf{V} \cdot d\mathbf{r} = \iint_{\mathcal{S}} (\nabla \times \mathbf{V}) \cdot \mathbf{n}\, dS$$

  where $\mathcal{C}$ is a great circle of the sphere and $\mathcal{S}$ is a hemisphere bounded by $\mathcal{C}$. The Flux field related to $\mathbf{V}$ is now

$$\begin{aligned}\nabla \times \mathbf{V} &= \nabla \times \big(\mathbf{V_1} + \mathbf{V_2}\big) \\ &= \nabla \times \big(-\nabla f + \nabla \times \mathbf{G}\big) \\ &= -\nabla \times \nabla f + \nabla \times (\nabla \times \mathbf{G}) \\ &= \nabla \times (\nabla \times \mathbf{G}) \\ &= \nabla \times \mathbf{V_2}\end{aligned}$$

  where we have used the fact that the flux of a conservative field is $\mathbf{0}$. From here we see that only the flux of the solenoidal/incompressible vector field, $\mathbf{V_2}$, makes a contribution to the flow around a great circle.

(b) From part (a) we know that we only need $\mathbf{V_2}$ for this calculation.

$$\begin{aligned}\mathbf{V_2} &= \nabla \times \mathbf{G} \\ &= \Big((0)_y - (xyz)_z\Big)\mathbf{i} + \Big((y^2+z^2)_z - (0)_x\Big)\mathbf{j} \\ &\quad + \Big((xyz)_x - (y^2+z^2)_y\Big)\mathbf{k} \\ &= -xy\mathbf{i} + 2z\mathbf{j} + (yz - 2y)\mathbf{k}\end{aligned}$$

There are now two option for calculating the flow.

- Option 1: Using Stoke's Theorem, we can calculate the flow around the great circle in the yz-plane, g_{yz}, by integrating the flux of $\mathbf{V_2}$ through the disc in the xy-plane bounded by this great circle, $\mathcal{D}$.

$$\begin{aligned}\nabla \times \mathbf{V_2} &= \Big((yz-2y)_y - (2z)_z\Big)\mathbf{i} + \Big((-xy)_z - (yz-2y)_x\Big)\mathbf{j} \\ &\quad + \Big((2z)_x - (-xy)_y\Big)\mathbf{k} \\ &= (z-4)\,\mathbf{i} + x\,\mathbf{k}\end{aligned}$$

Now,

$$\begin{aligned}
\oint_{g_{yz}} \mathbf{V_2} \cdot \mathbf{dr} &= \iint_{\mathcal{D}} (\nabla \times \mathbf{V_2}) \cdot \mathbf{dS} \\
&= \iint_{\mathcal{D}} \Big((z-4)\mathbf{i} + x\mathbf{k} \Big) \cdot \mathbf{i} \; dy\, dz \\
&= \iint_{\mathcal{D}} z - 4 \; dy\, dz \\
&= \iint_{\mathcal{D}} z \; dy\, dz - 4 \iint_{\mathcal{D}} 1 \; dy\, dz \\
&= \qquad 0 \qquad - 4 \left(6^2 \pi\right) \\
&= -144\pi
\end{aligned}$$

Here we have used both the fact that z is symmetric about the y-axis to deduce that the first integral is zero and the fact that the integral of 1 gives the area of the region to quickly calculate the second integral.

- Option 2: The flow can also be computed directly. The parameterization for the great circle in the yz-plane with a counterclockwise orientation is

$$\mathbf{r}(t) = \langle 0, 6\cos(t), 6\sin(t) \rangle. \tag{1}$$

and the vector field around the great circle is given by

$$\mathbf{V_2}\big(\mathbf{r}(t)\big) = \langle 0,\ 12\cos(t),\ 36\cos(t)\sin(t) - 12\cos(t) \rangle. \tag{2}$$

$$\begin{aligned}
\oint_{g_{yz}} \mathbf{V_2} \cdot \mathbf{dr} &= \int_0^{2\pi} \mathbf{V_2}\big(\mathbf{r}(t)\big) \cdot \mathbf{r}'(t)\, dt \\
&= \int_0^{2\pi} -144\cos(t)\sin(t) + 216\cos^2(t)\sin(t)\, dt \\
&= -144\pi
\end{aligned}$$

(c) There are two ways to think about this problem.

- Option 1: From the divergence theorem we know that the flux across the boundary is related to the divergence on the interior region,

$$\iint_S \mathbf{V} \cdot\ \mathbf{dS} = \iiint_{\mathcal{E}} \nabla \cdot \mathbf{V}\, dV.$$

Since $\mathbf{V_2}$ can be expressed as the curl of the vector field $\mathbf{G}$ we know $\mathbf{V_2}$ is incompressible (since $\nabla \cdot \mathbf{V}_2 = \nabla \cdot (\nabla \times \mathbf{G}) = 0$, $\mathbf{V}_2$ is incompressible, that is, $\nabla \cdot \mathbf{V_2} = 0$).
Thus, $\nabla \cdot \mathbf{V} = \nabla \cdot \mathbf{V_1}$ and only $\mathbf{V}_1$ contributes to the total outward flux.

- Option 2: By breaking the sphere into an upper hemi-sphere, U, lower hemisphere, L, divided by the great circle in the xy-plane, g, we can use Stoke's Theorem to find

$$\begin{aligned}
\iint_S \mathbf{V_2} \cdot\ \mathbf{dS} &= \iint_U \nabla \times \mathbf{G} \cdot\ \mathbf{dS} + \iint_L \nabla \times \mathbf{G} \cdot\ \mathbf{dS} && (3) \\
&= \oint_g \mathbf{G} \cdot \mathbf{dr} + \oint_g \mathbf{G} \cdot \mathbf{dr} && (4) \\
&= \oint_g \mathbf{G} \cdot \mathbf{dr} - \oint_g \mathbf{G} \cdot \mathbf{dr} && (5) \\
&= 0 && (6)
\end{aligned}$$

Notice that the direction of the first line integral in 4 is counterclockwise (since the normal direction on the upper hemisphere points up) and the direction of the second line integral is clockwise (since the normal direction on the lower hemisphere points down). Changing the direction of the second integral changes the sign of the integral, so the clockwise and counter-clockwise integrals cancel out. Thus, $\mathbf{V_2}$ does not contribute to the outward flux through the sphere.

(d) From part (c) we know that we only need $\mathbf{V_1}$ for this calculation.

$$\mathbf{V}_1 = -\nabla f = \langle 2x, 2y, -6z \rangle \tag{7}$$

Using the Divergence Theorem, we can calculate the flux through the sphere by integrating the divergence of $\mathbf{V_2}$ on the interior of the sphere, E.

$$\begin{aligned} \nabla \cdot \mathbf{V_1} &= \partial_x(2x) + \partial_y(2y) + \partial_z(-6z) && \text{(8a)} \\ &= -2 && \text{(8b)} \end{aligned}$$

Now,

$$\begin{aligned} \iint_S \mathbf{V_1} \cdot \mathbf{dS} &= \iiint_E \nabla \cdot \mathbf{V_1}\, dV && \text{(9a)} \\ &= \iiint_E -2\, dV && \text{(9b)} \\ &= -2 \iiint_E 1\, dV && \text{(9c)} \\ &= -2\left(\frac{4}{3}6^3\pi\right) && \text{(9d)} \\ &= -576\pi && \text{(9e)} \end{aligned}$$

Here we have used the fact that the triple integral of 1 gives the volume of the region.

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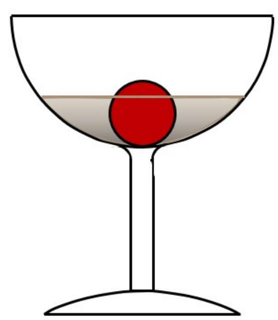
Problem 6 (15 pts)

A cocktail glass with a hemispherical bowl of radius 4 cm contains a cherry of radius 1 cm, positioned as drawn. Suppose the glass is filled to a depth of h cm, where $0 < h < 2$ (i.e., the cherry is partially submerged). Use integration in cylindrical coordinates with the order $dr\,dz\,d\theta$ to determine the volume of liquid (in terms of h) in the glass. *Hint:* Place the origin of your coordinate system at the bottom of the cherry.



SOLUTION:

The equation of the sphere representing the cherry is $x^2 + y^2 + (z-1)^2 = 1$, which can be expressed as $r^2 + (z-1)^2 = 1$ in cylindrical coordinates. Solving for r in terms of z gives

$$r = \sqrt{1-(z-1)^2} = \sqrt{2z - z^2}$$

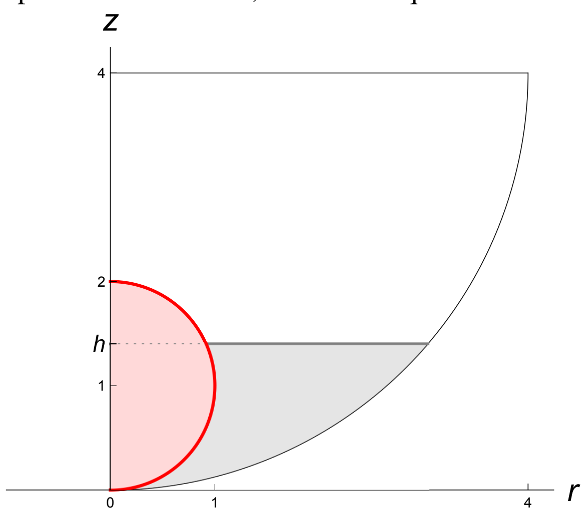
(we can ignore the negative root because we are in the first quadrant in the rz-plane)

Similarly, the equation of the sphere associated with the bowl is $x^2 + y^2 + (z-4)^2 = 16$, which can be expressed as $r^2 + (z-4)^2 = 16$ in cylindrical coordinates. Solving for r in terms of z gives

$$r = \sqrt{16 - (z-4)^2} = \sqrt{8z - z^2}$$

where we may ignore the negative root again.

The relevant picture in the rz-plane looks like this, where the liquid we are interested in is shaded gray:



We can evaluate the volume of the liquid corresponding to any value of h between 0 and 2 by integrating in cylindrical coordinates, as follows:

$$V = \int_0^{2\pi} \int_0^h \int_{\sqrt{2z-z^2}}^{\sqrt{8z-z^2}} r dr dz d\theta = 2\pi \int_0^h \frac{r^2}{2}\bigg|_{\sqrt{2z-z^2}}^{\sqrt{8z-z^2}} dz$$

$$= \pi \int_0^h \left[\left(8z - z^2\right) - \left(2z - z^2\right)\right] dz = \pi \int_0^h 6z dz = \boxed{3\pi h^2}$$

■

End Of Exam
