APPM 2350—Final Exam (Cumulative)
Wednesday May 5th, 10:30am-1pm 2021

Show all your work and simplify your answers. Answers with no justification will receive no points.

If you are asked to fully set-up but not evaluate integrals, this means to find the bounds of integration, fully simplify the integrands, and set-up in the coordinate system that leads to the simplest and fewest integrals. For line integrals and surface integrals, this also means correctly converting ds and dS.

Problem 1 (24 pts)

Consider a particle moving along a path $\mathcal{C}$ in space, where distance is measured in meters, time is measured in seconds, and the temperature at any point is given by the function $T(x, y, z)$ (in degrees Fahrenheit).

At a particular instant in time, t^* (and only at that instant in time), the position of the particle is $\mathbf{r}(t^*) = 3\mathbf{i} - 1\mathbf{j} + 2\mathbf{k}\ m$, its velocity is $\mathbf{v}(t^*) = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}\ \frac{m}{s}$ and its acceleration is $\mathbf{a}(t^*) = 2\mathbf{j}\ \frac{m}{s^2}$.

In addition, you are provided the following information about $T(x, y, z)$:
$T(2, 2, 1) = 10°\text{F}$ $\quad \nabla T(2, 2, 1) = 4\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}\ \frac{°F}{m}$

$T(3, -1, 2) = 6°\text{F}$ $\quad \nabla T(3, -1, 2) = -1\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}\ \frac{°F}{m}$

If possible, calculate each of the quantities below using only the information provided above. If there is not enough information provided, then write "Not enough info." (Note that you *may not need to use* all the information provided).

(a) At time $t = t^*$, what is the rate of change of the particle's temperature with respect to time t? *Include units.*
(b) If the particle continues along the path $\mathcal{C}$ for $\Delta t = 0.1$ seconds, by approximately how much will the temperature change? *Include units.*
(c) At time $t = t^*$, what is the rate of change of the particle's temperature with respect to distance traveled (arc length s) along its path? *Include units.*
(d) Calculate the curvature of the path at the position described. *Include units.*

Problem 2 (18 pts)

Water is flowing down a vertical cylindrical pipe of radius 3 inches, where the pipe is represented by the cylinder $x^2 + y^2 = 9$, bounded below by the xy-plane.

(a) Suppose the velocity vector field of the water at the outlet (i.e. the bottom opening) of the pipe is given by $\mathbf{v} = \langle 0, 0, -9 \rangle\ \frac{in}{min}$.
Find the total volume of water that flows out of the outlet of the pipe over 10 minutes. *Include units.*

(b) If the velocity vector field at the outlet of the pipe is instead given by $\mathbf{v} = \langle 0,\ 0,\ x^2 + y^2 - 9 \rangle\ \frac{in}{min}$, find the total volume of water that now flows out of the outlet of the pipe over 10 minutes. *Include units.*
(Fun fact: this velocity field models how water actually behaves when flowing through a pipe).

Problem 3 (15 pts)

A company produces three products. Suppose that the profit of the company, in millions of dollars, is $P(x, y, z) = 4x + 8y + 6z$, where x, y and z are the number of units (in thousands) of the three products sold. Manufacturing constraints force x, y and z to satisfy $x^2 + 4y^2 + 2z^2 \leq 800$. What is the maximum profit the company can earn? Provide full justification and simplify your answer. *Include units.*

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Problem 4 (38 pts)

Consider the force field

$$\mathbf{F} = xy\mathbf{i} - y\mathbf{j}$$

Suppose a closed, curved wire lies along the part of $x = 3 - y^2$ from $(2, -1)$ to $(-1, 2)$ followed by the line segment between $(-1, 2)$ and $(2, -1)$. The density at any point on the wire is proportional to the square of the distance from the origin at that point. Fully set up (but do not evaluate) integral(s) to find the quantities described below. **Please see the note at the top of the exam for a reminder of what we mean by fully set up**.

(a) Fully set up (do not evaluate) integral(s) to find the mass of the wire. Use k for your proportionality constant.

(b) Fully set up (do not evaluate) integral(s) to find the work done by **F** counterclockwise along the path of the wire via the following methods:

  (i) Setting up line integral(s).

  (ii) Using an appropriate theorem from Calculus 3 (again, just set up, do not actually evaluate).

Problem 5 (40 pts)

Let **V** be a vector field and let $\mathcal{S}$ be the surface of a sphere with radius 6 centered at the origin.
Helmholtz Theorem says that any vector field on a closed region (like a sphere) can be decomposed into two special component fields, such that
$\mathbf{V} = \mathbf{V_1} + \mathbf{V_2}$, where $\mathbf{V_1} = -\nabla f(x, y, z)$ and $\mathbf{V_2} = \nabla \times \mathbf{G}$.

Suppose $f(x, y, z) = 3z^2 - x^2 - y^2$ and $\mathbf{G} = \langle y^2 + z^2,\ xyz,\ 0\rangle$.

(a) Which vector field(s) ($\mathbf{V_1}$ and/or $\mathbf{V_2}$) contribute to the total flow of **V** around a great circle of the sphere? Explain your reasoning. (Reminder: a great circle is a circle formed by intersecting a sphere with a plane directly through the center of the sphere).

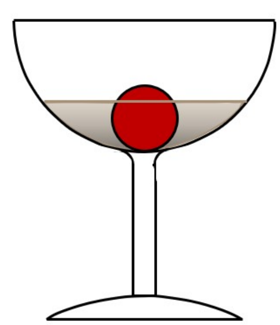
(b) Find the flow of **V** around the great circle of the sphere in the yz-plane, in a counterclockwise direction when viewed from the positive x-axis.

(c) Which vector field(s) ($\mathbf{V_1}$ and/or $\mathbf{V_2}$) contribute to the total flux of **V** through the surface of the sphere? Explain your reasoning.

(d) Find the total outward flux of **V** through the surface of the sphere.

Problem 6 (15 pts)

A cocktail glass with a hemispherical bowl of radius 4 cm contains a cherry of radius 1 cm, positioned as drawn. Suppose the glass is filled to a depth of h cm, where $0 < h < 2$ (i.e., the cherry is partially submerged). Use integration in cylindrical coordinates with the order $dr\,dz\,d\theta$ to determine the volume of liquid (in terms of h) in the glass. *Hint:* Place the origin of your coordinate system at the bottom of the cherry.



End Of Exam