APPM 2350—Exam 3 (Chapter 12 and 13A)
Wednesday April 7th, 7:30pm-9pm 2021

Show all your work and simplify your answers. Answers with no justification will receive no points. .

Problem 1 (24 points)

Consider the integral $\displaystyle\iint_{\mathcal{R}} (x^2+y^2)\ \mathrm{dA}$ where $\mathcal{R}$ is the region in the **first quadrant** given by

$$\mathcal{R} = \left\{ (x,y) \;\middle|\; 1 \le x \le \sqrt{2} \text{ and } x^2+y^2 \le 4 \text{ and } y \ge 0 \right\}.$$

(a) Sketch and shade the region $\mathcal{R}$ in the first quadrant of the xy-plane. Label any intercepts.

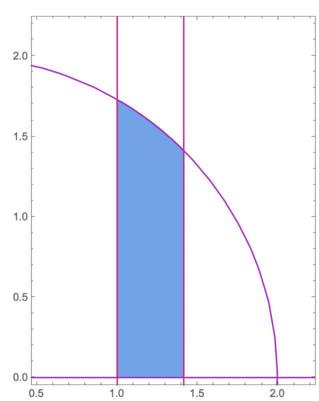
(b) Set up the double integral(s) in terms of x and y with order of integration $dy\,dx$. Do not evaluate.

(c) Set up the double integral(s) in terms of x and y with order of integration $dx\,dy$. Do not evaluate.

(d) Set up the double integral(s) in terms of polar coordinates with order of integration $dr\,d\theta$. Do not evaluate.

(e) Which version of the double integral would you rather compute and, more importantly, **why**? (Just explain which one you would prefer to actually evaluate, but do not evaluate the integral).

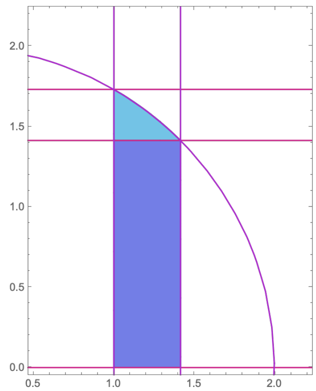
<u>**SOLUTION:**</u>

(a) 

(b) When integrating y first, the lower bound is the x-axis and the upper bound is the circle of radius 2.

$$\int_1^{\sqrt{2}} \int_0^{\sqrt{4-x^2}} (x^2+y^2)\,dy\,dx \tag{1}$$

(c) When integrating x first, the region of integration must be broken into two regions. The first region is the rectangle $[1,\sqrt{2}]\times[0,\sqrt{2}]$. The second region is bounded on the left by the line $x=1$ and on the right by the circle $x=\sqrt{4-y^2}$.

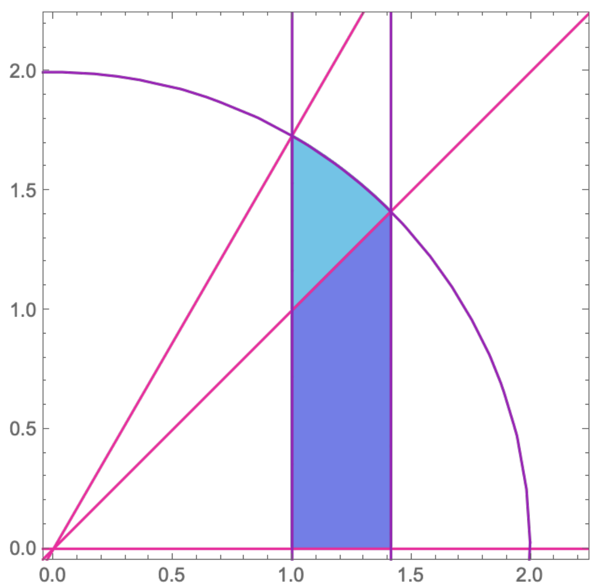


$$\int_0^{\sqrt{2}}\int_1^{\sqrt{2}}(x^2+y^2)dx\,dy+\int_{\sqrt{2}}^{\sqrt{3}}\int_1^{\sqrt{4-y^2}}(x^2+y^2)dx\,dy \quad (2)$$

(d) When integrating in polar coordinates, the region must be broken into two pieces. For the first region, the inner boundary for r is the line $x=1$. In polar coordinate this line becomes $r\cos\theta=1$ or $r=\sec\theta$. Similarly, the outer boundary is the line $r=\sqrt{2}\sec\theta$. The cut off angle for the first region occurs when the circle intersects line $\sqrt{2}\sec\theta=2$, which gives an angle of $\pi/4$.

In the second region goes from $\theta=\pi/4$ to $\theta=\pi/3$ where the circle meets the line $r=\sec\theta$.

$$\int_0^{\frac{\pi}{4}}\int_{\sec(\theta)}^{\sqrt{2}\sec(\theta)} r^3dr\,d\theta+\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\int_{\sec(\theta)}^{2} r^3dr\,d\theta \quad (3)$$



(e) There's multiple ways to answer this question. You could argue that the integral from part (b) is the easiest to compute since this is the only version that only uses one region. Alternatively, you could argue that the integral from part (d) is the easiest to compute since the integrand is radially symmetric as is the circular boundary.

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Problem 2 (24 pts)

Use an appropriate uv-transformation to evaluate the integral

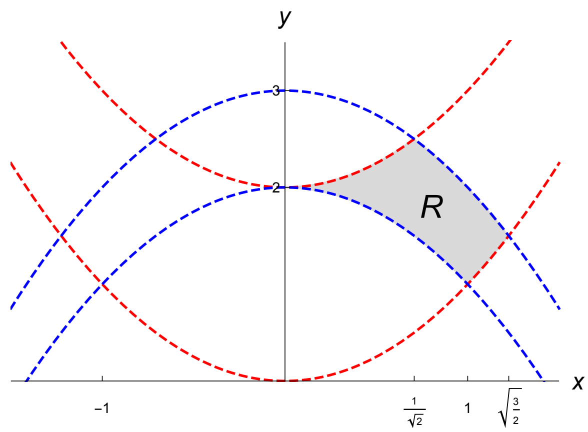
$$\iint_{\mathcal{R}} xe^{y-x^2}\,dA$$

where $\mathcal{R}$ is the region in the **first quadrant** bounded by $y=x^2$, $y=x^2+2$, $y=2-x^2$, and $y=3-x^2$.

Be sure to include a sketch of $\mathcal{R}$ and a sketch of the corresponding region S in the uv-plane in your solution.

SOLUTION:

The region R looks like this:



Think of the given curves as $y - x^2 = 0$, $y - x^2 = 2$, $y + x^2 = 2$ and $y + x^2 = 3$.

Setting $u = y - x^2$ and $v = y + x^2$ (which makes sense in light of the $y - x^2$ in the integrand), we see that

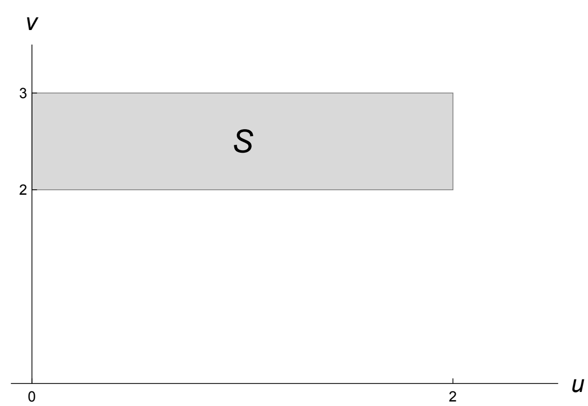
$$u + v = 2y \implies y = \frac{1}{2}(u + v)$$

and

$$v - u = 2x^2 \implies x^2 = \frac{1}{2}(v - u) \implies |x| = \frac{1}{\sqrt{2}}\sqrt{v - u} \implies x = \frac{1}{\sqrt{2}}\sqrt{v - u}$$

because we're given that $x \geq 0$.

Also, note that $0 \leq u \leq 2$ and $2 \leq v \leq 3$ so the region S in the uv-plane is a rectangle:



The Jacobian of the transformation is

$$J(u, v) = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \det \begin{bmatrix} \frac{1}{\sqrt{2}} \frac{-1}{2\sqrt{v-u}} & \frac{1}{\sqrt{2}} \frac{1}{2\sqrt{v-u}} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$= \frac{-1}{4\sqrt{2}\sqrt{v - u}} - \frac{1}{4\sqrt{2}\sqrt{v - u}}$$

$$= \frac{-1}{2\sqrt{2}\sqrt{v - u}}$$

Thus

$$\begin{aligned}
\iint_R xe^{y-x^2}\,dA &= \iint_S \left(\frac{1}{\sqrt{2}}\sqrt{v-u}\right) e^u \left|\frac{-1}{2\sqrt{2}\sqrt{v-u}}\right| \,dA_{uv} \\
&= \frac{1}{4}\int_2^3 \int_0^2 e^u \,du\,dv \\
&= \frac{1}{4}\int_2^3 \left[e^u\right]_0^2 \,dv \\
&= \frac{1}{4}\int_2^3 (e^2-1)\,dv \\
&= \frac{1}{4}(e^2-1)\cdot 1 \\
&= \frac{1}{4}(e^2-1)
\end{aligned}$$

Note. Other substitutions are possible, like $u = x$ and $v = y - x^2$

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Problem 3 (36 points)

Consider the following 2 spheres:

$$\text{Sphere A:}\quad x^2+y^2+z^2=2$$
$$\text{Sphere B:}\quad x^2+y^2+(z-1)^2=1$$

Let $\mathcal{E}$ be the intersection of the regions inside both spheres (i.e., the 3D solid that's common to both spheres). Suppose the temperature at any point in space is given by $T(x,y,z) = x^2y^2z^2$ degrees Fahrenheit.

(a) Sketch and shade the cross section of $\mathcal{E}$ in the rz-plane. Label any intercepts.

(b) Sketch and shade the projection of $\mathcal{E}$ onto the xy-plane. Label any intercepts.

(c) Set up integral(s) to find the **volume of** $\mathcal{E}$ using spherical coordinates in the order $d\rho\,d\phi\,d\theta$. Do not evaluate the integral(s).

(d) Set up integral(s) to find the **z component of the centroid of** $\mathcal{E}$ using cylindrical coordinates in the order $dz\,dr\,d\theta$. Do not evaluate the integral(s).

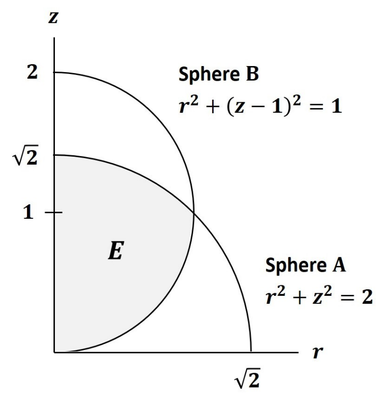
(e) Set up integral(s) to find the **average temperature on the surface of the top hemisphere of Sphere A** (use the coordinate system that leads to the fewest number and simplest integrals). Do not evaluate the integral(s).

SOLUTION:

(a) Sketch and shade the cross section of $\mathcal{E}$ in the rz-plane. Label any intercepts.

Since $x^2 + y^2 = r^2$, the equations of Spheres A and B can be expressed as $r^2 + z^2 = 2$ and $r^2 + (z-1)^2 = 1$, respectively. The equation associated with Sphere A represents a circle of radius $\sqrt{2}$ centered at $(r,z) = (0,0)$, and the equation associated with Sphere B represents a circle of radius 1 centered at $(r,z) = (0,1)$.

The rz-plane does not include negative values of r, since that variable is a measure of distance. Furthermore, since Sphere B is located entirely on or above the xy-plane, no point in the region of intersection $\mathcal{E}$ has a negative z value. Therefore, only the first quadrant of the rz-plane needs to be depicted, as follows:



(b) Sketch and shade the projection of $\mathcal{E}$ onto the xy-plane. Label any intercepts.

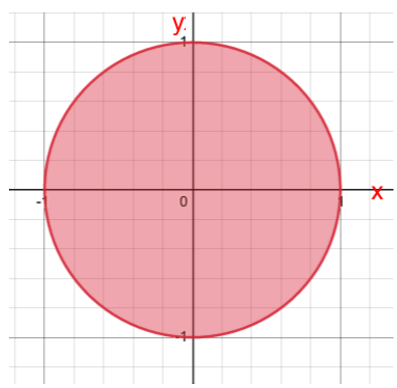
The first step is to identify the intersection of the two spheres in the rz-plane, which entails solving the equations of the two spheres simultaneously. The equation associated with Sphere B can be subtracted from the equation associated with Sphere A as the first step in the following solution:

$$z^2 - (z-1)^2 = 2 - 1 \quad \Rightarrow \quad z^2 - (z^2 - 2z + 1) = 1 \Rightarrow \quad 2z = 2 \quad \Rightarrow \quad z = 1$$

$$r^2 + 1^2 = 2 \quad \Rightarrow \quad r = 1 \qquad \text{(take only the positive root since } r \text{ must be positive)}$$

Therefore, the two spheres intersect at the point $(r, z) = (1, 1)$ in the rz-plane.

Thus, the projection of $\mathcal{E}$ onto the xy-plane is a circle of radius 1, centered at the origin.



(c) Set up integral(s) to find the **volume of** $\mathcal{E}$ using spherical coordinates in the order $d\rho\, d\phi\, d\theta$. Do not evaluate the integral(s).

$$V = \iiint_{\mathcal{E}} dV$$

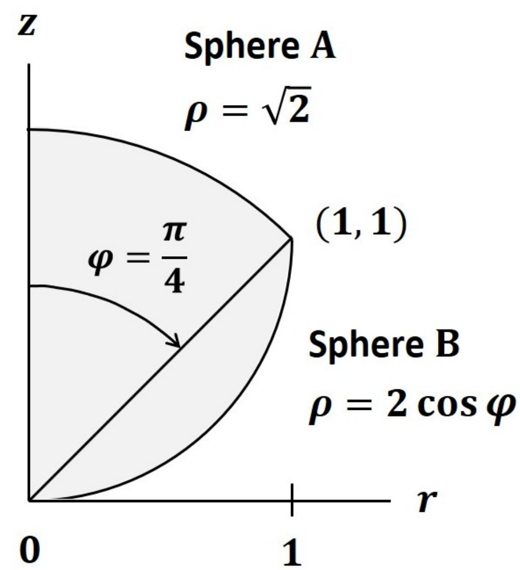
Since $x^2 + y^2 + z^2 = \rho^2$, the equation of Sphere A can be expressed as $\rho^2 = 2$, which implies that $\rho = \sqrt{2}$.

The equation of Sphere B can be re-expressed as follows:

$$x^2 + y^2 + (z-1)^2 = 1 \quad \Rightarrow \quad x^2 + y^2 + z^2 - 2z + 1 = 1 \quad \Rightarrow \quad \rho^2 = 2z = 2\rho\cos\phi$$

$$\Rightarrow \quad \rho = 2\cos\phi$$

Also, the line segment between the origin and the point $(1, 1)$ in the rz-plane has a slope of 1, which implies that it forms a 45-degree angle with the positive z-axis. Therefore, the equation of that line segment is $\phi = \frac{\pi}{4}$, as indicated in the following diagram:



Integration over region $\mathcal{E}$ in spherical coordinates would require two triple integrals: one associated with $0 \leq \phi \leq \dfrac{\pi}{4}$ and one associated with $\dfrac{\pi}{4} \leq \phi \leq \dfrac{\pi}{2}$. In the first triple integral, ρ runs from 0 to $\sqrt{2}$, and in the second triple integral, ρ runs from 0 to $2\cos\phi$.

Therefore, the following expression represents the volume of $\mathcal{E}$:

$$\boxed{V = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{2}} \rho^2 \sin\phi d\rho d\phi d\theta + \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{2\cos\phi} \rho^2 \sin\phi d\rho d\phi d\theta}$$

(d) Set up integral(s) to find the **z component of the centroid of** $\mathcal{E}$ using cylindrical coordinates in the order $dz\,dr\,d\theta$. Do not evaluate the integral(s).

$$\bar{z} = \frac{\iiint\limits_{\mathcal{E}} z\,dV}{\iiint\limits_{\mathcal{E}} dV}$$

The solid region $\mathcal{E}$ is obtained by revolving its cross section in the rz-plane from $\theta = 0$ to $\theta = 2\pi$. For each such value of θ, values of r between 0 to 1 must be considered. The next step is to identify the limits of z for each such value of r.

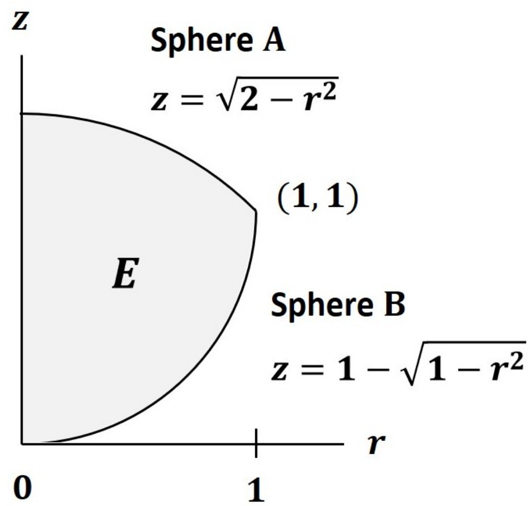
The equation associated with Sphere A can be re-expressed as follows:

$$r^2 + z^2 = 2 \quad \Rightarrow \quad z = \sqrt{2 - r^2} \qquad (z \geq 0 \text{ on region } E)$$

Similarly, the equation associated with Sphere B can be re-expressed as follows:

$$r^2 + (z-1)^2 = 1 \quad \Rightarrow \quad z = 1 \pm \sqrt{1 - r^2}$$

The figure from part (a) indicates that the portion of Sphere B that forms a boundary of region $\mathcal{E}$ is its **lower** hemisphere, which is associated with the **negative** root in the preceding expression. Therefore, the expression for the lower portion of $\mathcal{E}$ is $z = 1 - \sqrt{1-r^2}$. The following diagram is a modification of the diagram from part (a):



The following triple integral for the volume of $\mathcal{E}$ is closely related to the preceding diagram:

$$V = \int_0^{2\pi} \int_0^1 \int_{1-\sqrt{1-r^2}}^{\sqrt{2-r^2}} r dz dr d\theta$$

Thus

$$\boxed{\bar{z} = \frac{\int_0^{2\pi} \int_0^1 \int_{1-\sqrt{1-r^2}}^{\sqrt{2-r^2}} zr\, dz\, dr\, d\theta}{\int_0^{2\pi} \int_0^1 \int_{1-\sqrt{1-r^2}}^{\sqrt{2-r^2}} r\, dz\, dr\, d\theta}}$$

(e) Set up integral(s) to find the **average temperature on the surface of the top hemisphere of Sphere A** (use the coordinate system that leads to the fewest number and simplest integrals). Do not evaluate the integral(s).

$$T_{ave} = \frac{\iint\limits_S T(x,y,z)\, dS}{\iint\limits_S dS}$$

The denominator, $\iint\limits_S dS$, is equal to the surface area of the hemisphere, which is $\frac{1}{2} 4\pi(\sqrt{2})^2 = 4\pi$.

Thus

$$T_{ave} = \frac{1}{4\pi} \iint\limits_S x^2 y^2 z^2\, dS$$

Let $g(x,y,z) = x^2 + y^2 + z^2$

It's easiest to project the surface onto the xy-plane (otherwise you will have to use 2 surface integrals). Thus, let $\hat{p} = \mathbf{k}$.

$$dS = \frac{||\nabla g||}{|\nabla g \cdot \mathbf{k}|}\, dA = \frac{2\sqrt{x^2+y^2+z^2}}{|2z|}\, dA = \frac{2\sqrt{2}}{2z}\, dA = \frac{\sqrt{2}}{z}\, dA$$

since $z \geq 0$ on the top hemisphere.

$$T_{ave} = \frac{1}{4\pi}\iint\limits_S x^2y^2z^2\,dS = \frac{1}{4\pi}\iint\limits_R x^2y^2z^2\frac{\sqrt{2}}{z}\,dA$$

where R is the circle in the xy-plane with radius $\sqrt{2}$ centered at the origin.
Since on the top hemisphere $z = \sqrt{2-x^2-y^2}$, this integral becomes:

$$T_{ave} = \frac{1}{4\pi}\iint\limits_R \sqrt{2}x^2y^2\sqrt{2-x^2-y^2}\,dA$$

Given the region of integration and the integrand, this is easiest to evaluate in polar coordinates:

$$T_{ave} = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\sqrt{2}} \sqrt{2}(r\cos\theta)^2(r\sin\theta)^2(\sqrt{2-r^2})r\,dr\,d\theta$$

$$\boxed{= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\sqrt{2}} \sqrt{2}r^5\cos^2\theta\sin^2\theta\sqrt{2-r^2}\,dr\,d\theta}$$

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Problem 4 (16 points)

The following questions are not related:

(a) The following double integral gives the volume of a 3D region $\mathcal{E}$. Sketch and shade the region $\mathcal{E}$ in xyz-space, and label your axes and any intercepts. Do not evaluate the integral.

$$\int_0^3\int_0^y 4\ dx\,dy$$

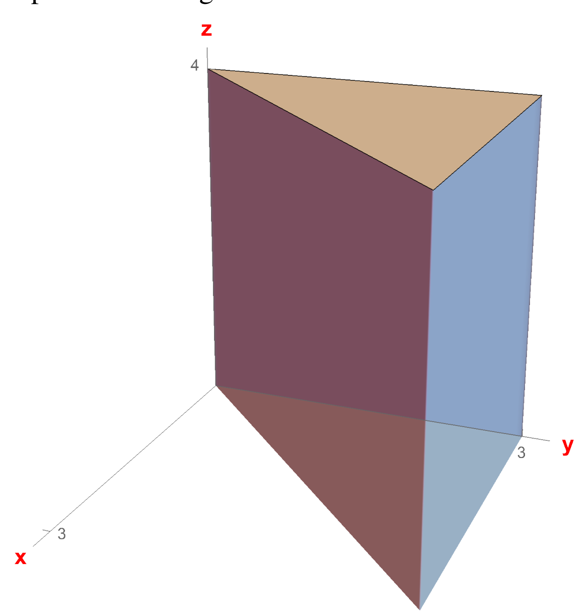
(b) Given

$$\int_0^2\int_0^{y/2}\int_5^{9-y^2} e^{3z}(y+2x)\,dz\,dx\,dy$$

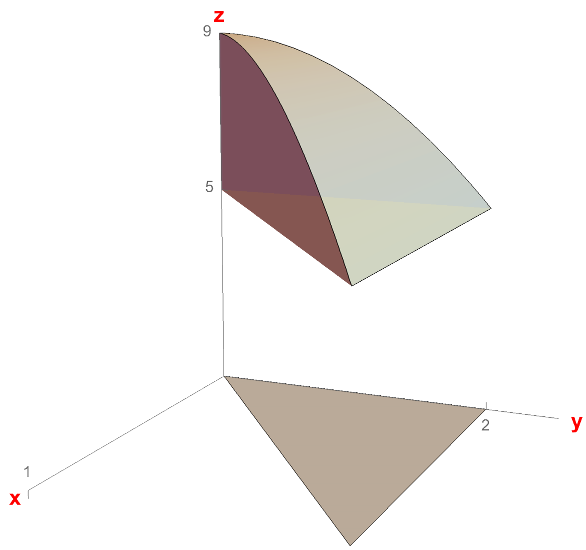
Rewrite as equivalent integral(s) using the ordering $dx\,dy\,dz$. Do not evaluate.

<u>SOLUTION:</u>

(a) The region is the triangular prism with height 4 shown below:



(b) The region of integration is given by:

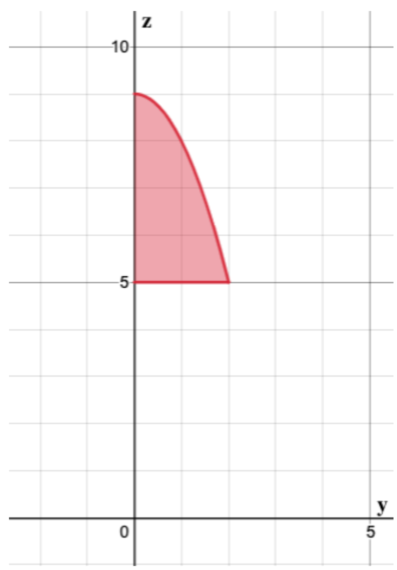


Using this region we can set-up bounds using the new ordering: $dx\,dy\,dz$ as follows:

x bounds: Shoot an arrow through the region parallel to the x-axis. The arrow enters along $x = 0$ and exits along the plane $x = y/2$

To find the y and z bounds, we look at the projection of the region onto the yz-plane.

This projection is the region shown here:



Shoot an arrow parallel to the y-axis (i.e. horizontal in this case). The arrow enters the region along $y = 0$ and exits along $y = \sqrt{9-z}$.

Finally the outside z limits are the largest and smallest values of z on the region: $z = 5$ to $z = 9$.

The integrand is not affected by the change in integration order.

Thus our final answer is:

$$\boxed{\int_5^9 \int_0^{\sqrt{9-z}} \int_0^{y/2} e^{3z}(y+2x)\,dx\,dy\,dz}$$

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End Of Exam