APPM 2350—Exam 2

Wednesday March 10th, 7:30pm-9pm 2021

Show all your work and simplify your answers. Answers with no justification will receive no points. You are allowed one 8.5×11-in page of notes (ONE side). You must turn this in with your exam at the end. You may NOT use a calculator, smartphone, smartwatch, the Internet or any other electronic device.

Problem 1 (28 pts)
Consider the function $f(x, y) = e^y(x^2 - y^2)$

(a) Find the quadratic (i.e. 2nd order) Taylor approximation of $f(x, y)$ centered at $(3, 0)$. (Tip: Double-check your work finding f_x and f_y before moving on, as the rest of the problem depends on these).
(b) Use Taylor's Formula to find a reasonable upper bound on the error of this quadratic approximation if $|x - 3| \leq 0.1$ and $|y| \leq 0.1$. (You can leave your answer unsimplified).
(c) Identify and classify all critical points of $f(x, y)$.

SOLUTION:

(a) We begin by finding the first- and second-order partial derivatives of f:

$$\begin{aligned}
f_x(x, y) &= 2xe^y \\
f_y(x, y) &= e^y(-2y) + e^y(x^2 - y^2) = (x^2 - y^2 - 2y)e^y \\
f_{xx}(x, y) &= 2e^y \\
f_{xy}(x, y) &= 2xe^y \\
f_{yy}(x, y) &= e^y(-2y - 2) + e^y(x^2 - y^2 - 2y) = (x^2 - y^2 - 4y - 2)e^y
\end{aligned}$$

The general form of the quadratic Taylor approximation about the point $(3, 0)$ is

$$\begin{aligned}
Q(x, y) = &f(3, 0) + f_x(3, 0)(x - 3) + f_y(3, 0)y \\
&+ \frac{1}{2}\left[f_{xx}(3, 0)x^2 + 2f_{xy}(3, 0)(x - 3)y + f_{yy}(3, 0)y^2\right]
\end{aligned}$$

The Taylor coefficients are evaluated as follows:

$$\begin{aligned}
f(3, 0) &= e^0(3^2 - 0^2) = 9 \\
f_x(3, 0) &= (2)(3)e^0 = 6 \\
f_y(3, 0) &= [3^2 - 0^2 - (2)(0)]e^0 = 9 \\
f_{xx}(3, 0) &= (2)e^0 = 2 \\
f_{xy}(3, 0) &= (2)(3)e^0 = 6 \\
f_{yy}(3, 0) &= [3^2 - 0^2 - (4)(0) - 2]e^0 = 7
\end{aligned}$$

Therefore, the quadratic Taylor approximation, $Q(x, y)$, of $f(x, y)$ centered at $(3, 0)$ is:

$$\begin{aligned}
Q(x, y) &= 9 + 6(x - 3) + 9y + \frac{1}{2}\left[2(x - 3)^2 + (2)(6)(x - 3)y + 7y^2\right] \\
&= \boxed{9 + 6(x - 3) + 9y + (x - 3)^2 + 6(x - 3)y + \frac{7}{2}y^2}
\end{aligned}$$

(b) To derive an upper bound on the error of the quadratic approximation, the third-order partial derivatives must be evaluated:

$$f_{xxx}(x,y) = 0$$
$$f_{xxy}(x,y) = 2e^y$$
$$f_{xyy}(x,y) = 2xe^y$$
$$f_{yyy}(x,y) = (x^2 - y^2 - 4y - 2)e^y + (-2y-4)e^y = (x^2 - y^2 - 6y - 6)e^y$$

On the region $\{(x,y) : |x-3| \le 0.1, |y| \le 0.1\}$, the following upper bounds can be placed on the magnitudes of the third-order partial derivatives:

$$|f_{xxx}| = 0$$
$$|f_{xxy}| \le 2e^{0.1}$$
$$|f_{xyy}| \le (2)(3.1)e^{0.1} = 6.2e^{0.1}$$
$$|f_{yyy}| = |(x^2 - y^2 - 6y - 6)e^y| \le \left(|x^2| + |-y^2| + |-6y| + 6\right) e^y \quad \text{(Triangle Inequality)}$$
$$= \left[(3.1)^2 + (0.1)^2 + (6)(0.1) + 6\right] e^{0.1} = 16.22e^{0.1}$$

Note that the error bound derived through the preceding application of the Triangle Inequality is conservative. The least upper bound for $|f_{yyy}|$ is actually $4.2e^{-0.1}$ (see below for more explanation) rather than $16.22e^{0.1}$, **which implies that the least upper bound for M is the value of $6.2e^{0.1}$ that was derived for $|f_{xyy}|$.**

For the purpose of this exam we will accept any value of M between $6.2e^{0.1}$ and $16.22e^{0.1}$ as long as sufficient justification is given.

$$|E(x,y)| \le \frac{M}{3!}\left[|x-3| + |y|\right]^3 = \frac{6.2e^{0.1}}{6}(0.1+0.1)^3 = \boxed{\frac{(6.2)(0.2)^3e^{0.1}}{6}}$$

Specifically, the maximum value of $4.2e^{-0.1}$ for $|f_{yyy}|$ is attained at the point $(3.1, -0.1)$, which lies on a corner of the region of interest, $\{(x,y) : |x-3| \le 0.1, |y| \le 0.1\}$. However, the identification of $|f_{yyy}(3.1, -0.1)|$ as the absolute maximum value of $|f_{yyy}|$ on the region requires thorough analysis. Specifically, in addition to the four corner points of the region, the potential existence of relative maxima in the interior of the region and in the interior of each boundary line segment must be considered prior to concluding that $|f_{yyy}(3.1, -0.1)| = 4.2e^{-0.1}$ is the absolute maximum value.

(c) The critical points are located where both first-order partial derivatives equal zero. In other words, each critical point satisfies both of the following equations:

$$f_x(x,y) = 2xe^y = 0$$
$$f_y(x,y) = (x^2 - y^2 - 2y)e^y = 0$$

The first equation can only be satisfied by $x = 0$. Plugging that condition into the second equation implies that $y^2 + 2y = 0$, so that $y = 0$ or $y = -2$. Therefore, the two critical points are:

$$\boxed{(0,0), (0,-2)}$$

To classify the critical points, apply the Second Derivative Test. The metric used in the test is:

$$D(x,y) = [f_{xx}(x,y)][f_{yy}(x,y)] - [f_{xy}(x,y)]^2$$

The expressions for the second-order partial derivatives that were derived in part (a) are evaluated at the critical points as follows:

For the critical point $(0, 0)$, we have:

$$f_{xx}(0,0) = 2e^0 = 2$$
$$f_{xy}(0,0) = (2)(0)e^0 = 0$$
$$f_{yy}(0,0) = [0^2 - 0^2 - (4)(0) - 2]e^0 = -2$$
$$D(0,0) = (2)(-2) - (0)^2 = -4 < 0 \quad \Rightarrow \quad \boxed{\text{Saddle point at } (0,0)}$$

For the critical point $(0, -2)$, we have:

$$f_{xx}(0,-2) = 2e^{-2}$$
$$f_{xy}(0,-2) = (2)(0)e^{-2} = 0$$
$$f_{yy}(0,-2) = [0^2 - (-2)^2 - (4)(-2) - 2]e^{-2} = 2e^{-2}$$
$$D(0,-2) = (2e^{-2})(2e^{-2}) - (0)^2 = 4e^{-4} > 0 \quad \Rightarrow \quad \text{Relative extremum at } (0,-2)$$
$$f_{xx}(0,-2) = 2e^{-2} > 0 \quad \Rightarrow \quad \boxed{\text{Relative minimum at } (0,-2)}$$

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Problem 2 (18 points)
A beetle is walking along a path $\mathbf{r}(t)$ formed by the intersection of two rock surfaces. One surface, a curved vertical wall, is described by $y = \frac{1}{2}x^2$ and the other surface is a plane described by $z = 3x + \frac{1}{2}y$. (Ground level would be the xy-plane). The beetle starts at the origin and moves along the path towards the point P given by $(x, y, z) = (2, 2, 7)$. The temperature outside that morning is given by $T(x, y, z) = xyz + 20$.

(a) Determine a parameterization of the beetle's path, $\mathbf{r}(t)$, where t represents time.
(b) As the beetle passes over the point P, find the instantaneous rate of change of the temperature *with respect to time*, given your particular parameterization.
(c) As the beetle passes over the point P, calculate the instantaneous rate of change of the temperature with respect to *distance traveled* (in the direction of the beetle's path).

<u>**SOLUTION:**</u>

(a) There are infinite number of possible parameterizations.

Easiest: $\boxed{\mathbf{r}(t) = \left\langle t, \frac{t^2}{2}, 3t + \frac{t^2}{4} \right\rangle}$

(b) Using the chain rule,

$$\boxed{\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt}}$$

Given the paramterization chosen in part(a), at the point P, $t = 2$.

$$\nabla T = \langle yz, xz, xy \rangle$$
$$\implies \nabla T(2,2,7) = \langle 14, 14, 4 \rangle$$
$$\mathbf{r}'(t) = \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle = \langle 1, t, 3 + \frac{t}{2} \rangle$$
$$\implies \mathbf{r}'(2) = \langle 1, 2, 4, \rangle$$

Thus, given the parameterization in part (a), the answer is

$$\boxed{\frac{dT}{dt} = 14(1) + 14(2) + 4(4) = 58}$$

(Important Note: answers to this problem may vary as $\mathbf{r}'(t)$ depends on the specific parameterization chosen in part (a))

(c) **Method 1**:
$\frac{dT}{ds} = D_u T$ in the direction the beetle is traveling .
At the point $t = 2$, the beetle is traveling in the direction $\mathbf{r}'(2) = \langle 1, 2, 4 \rangle$.
Thus, let $\mathbf{u} = \frac{\mathbf{r}'(2)}{||\mathbf{r}'(2)||} = \frac{\langle 1,2,4 \rangle}{\sqrt{21}}$
Thus $\boxed{\frac{dT}{ds} = D_u T = \nabla T(2, 2, 7) \cdot \mathbf{u} = \langle 14, 14, 4 \rangle \cdot \frac{\langle 1, 2, 4 \rangle}{\sqrt{21}} = \frac{58}{\sqrt{21}}}$

Method 2: Using the chain rule,

$$\frac{dT}{ds} = \frac{dT}{dt}\frac{dt}{ds}$$

Where $\frac{ds}{dt} = ||\mathbf{r}'(t)||$
Thus

$$\boxed{\frac{dT}{ds} = \frac{dT}{dt}\frac{1}{||\mathbf{r}'(t)||}}$$

When $t = 2$,

$$\boxed{\frac{dT}{ds} = (58)(\frac{1}{\sqrt{21}})}$$

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Problem 3 (18 pts) Ignoring air resistance, the height of a rocket t seconds after launch and with thrust $a\ ft/s^2$ can be modeled by $h(t, a) = \frac{1}{2}(a - 32)t^2$ feet. Limited fuel capacity imposes the constraint $a^2 t = 10,000$, where a is once again the thrust and t the amount of time this thrust can be maintained before running out of fuel.

(a) Use the method of Lagrange multipliers to find the value of a that maximizes the height the rocket can reach. Simplify your answer.
(b) For the value of a you found in part (a), how long will the engines burn before running out of fuel? Leave your answer unsimplified.

<u>**SOLUTION:**</u>
(a) We want to maximize the function $h(t, a)$ subject to the constraint $a^2 t = 10000$. Setting $g(t, a) = a^2 t$, we need to solve the system

$$\nabla h(t, a) = \lambda \nabla g(t, a)$$
$$g(t, a) = 10000$$

or

$$\left\langle (a - 32)t, \frac{1}{2}t^2 \right\rangle = \langle \lambda a^2, 2\lambda a t \rangle$$
$$a^2 t = 10000$$

or

$$(a - 32)t = \lambda a^2$$
$$\frac{1}{2}t^2 = 2\lambda a t$$
$$a^2 t = 10000$$

The second equation can be rewritten

$$\frac{1}{2}t^2 - 2\lambda a t = 0 \qquad \text{or} \qquad \frac{1}{2}t(t - 4\lambda a) = 0$$

so that either $t = 0$ or $t = 4\lambda a$. But $t = 0$ doesn't satisfy the constraint, so t must equal $4\lambda a$. Note that $\lambda \neq 0$ and $a \neq 0$, for otherwise t would be zero again, which is impossible.
Plugging $t = 4\lambda a$ in to the first equation yields

$$\begin{aligned} (a-32)4\lambda a &= \lambda a^2 \\ 4(a-32) &= a && \text{cancel } a \text{ and } \lambda\text{, neither of which is zero} \\ 3a &= 128 \\ a &= \frac{128}{3}\ ft/s^2 \end{aligned}$$

(ALTERNATE SOLUTION: realizing $t \neq 0$ and $a \neq 0$ due to the constraint, solve the first two equations for λ and set equal)
(b) Just use the constraint to solve for t:

$$\left(\frac{128}{3}\right)^2 t = 10000 \implies t = \frac{10000}{\left(\frac{128}{3}\right)^2}\ s$$

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Problem 4 (18 points)
Consider the function $g(x, y) = \frac{3x+2y}{\sqrt{x^2+y^2}}$

(a) Sketch the graph of the level curve of $g(x, y)$ that passes through the point $(-2, 3)$. Label axes and the value of g on the level curve.
(b) Prove algebraically that the following limit does not exist: $\lim_{(x,y)\to(0,0)} \frac{3x+2y}{\sqrt{x^2+y^2}}$

<u>**SOLUTION:**</u>

(a) The level curve that passes through the point $(-2, 3)$ is $g(x, y) = 0$, since the $g(-2, 3) = 0$. The equation for this level curve is

$$\begin{aligned} \frac{3x+2y}{\sqrt{x^2+y^2}} &= 0 \\ 3x + 2y &= 0 \\ y &= -\frac{3}{2}x, \quad (x, y) \neq (0, 0) \end{aligned}$$

See figure 1.

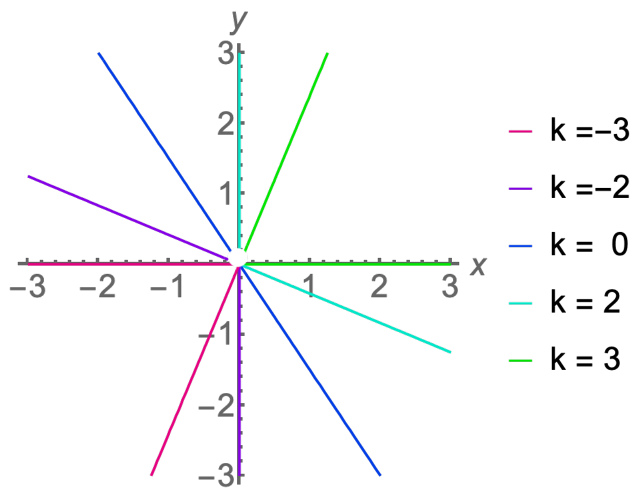

FIGURE 1. Contour plot for the function $g(x, y)$.

(b) To show that the limit does not exist we need to show that the 1D limit takes on different values for different paths through he point $(0, 0)$. For example, consider a straight line through $(0, 0)$ with an arbitrary slope

$$y = mx.$$

Making this substitution into $g(x, y)$ gives

$$\begin{aligned} \lim_{x\to 0^+} g(x, mx) &= \lim_{x\to 0^+} \frac{3x + 2mx}{\sqrt{x^2 + (mx)^2}} \\ &= \lim_{x\to 0^+} \frac{x(3 + 2m)}{x\sqrt{1 + m^2}} \\ &= \lim_{x\to 0^+} \frac{3 + 2m}{\sqrt{1 + m^2}} \\ &= \frac{3 + 2m}{\sqrt{1 + m^2}} \end{aligned}$$

Here, we see that the value of the limit depends on the slope of the path through the point $(0, 0)$, and therefore the limit does not exist. Also, note that we are taking the limit from the right, since the limit from the right will have the opposite sign. In fact, even the limit along a single line through $(0, 0)$ does not exist because the $\lim_{x\to 0^+} g(x, mx) \neq \lim_{x\to 0^-} g(x, mx)$.

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Problem 5 (18 pts) Let the height of a mountain be given by the function $H(x, y)$, with $H(1, 3) = 5,000$ ft and $\nabla H(1, 3) = 2\mathbf{i} - 7\mathbf{j}$

(a) You are standing on the mountain at the point $(1, 3, 5000)$ and you decide to hike in the steepest direction possible. Find a parametric equation for a line (in x, y and z) that is tangent to your path in this direction at the point $(1, 3, 5000)$

(b) Find the equation of the plane that's tangent to the mountain at the point $(x, y, z) = (1, 3, 5000)$. Give your answer in **standard form**, fully simplified.

SOLUTION:

(a) The gradient points in the direction of steepest ascent, so a unit vector in that direction is given by

$$\hat{\mathbf{u}} = \frac{\nabla H}{\|\nabla H\|}.$$

To find the z component of a vector tangent to the surface $z = H(x, y)$ in the direction $\hat{\mathbf{u}}$ we need the directional derivative

$$\frac{\mathrm{d}H}{\mathrm{d}\hat{\mathbf{u}}} = \nabla H \cdot \frac{\nabla H}{\|\nabla H\|} = \|\nabla H\|.$$

All together this gives a tangent vector of

$$\left\langle \frac{\frac{\partial H}{\partial x}}{\|\nabla H\|}, \frac{\frac{\partial H}{\partial y}}{\|\nabla H\|}, \|\nabla H\| \right\rangle$$

To simplify the algebra we can scale all terms by $\|\nabla H\|$ since the magnitude of the tangent vector doesn't matter.

$$\mathbf{V} = \left\langle \frac{\partial H}{\partial x}, \frac{\partial H}{\partial y}, \|\nabla H\|^2 \right\rangle$$

Now, the equation of a tangent line is give by

$$\mathbf{r}(t) = \mathbf{V}t + \mathbf{p}_0$$

where $\mathbf{p}_0$ is the point of tangency, $\langle 1, 3, 5000\rangle$. Plugging in $\frac{\partial H}{\partial x} = 2$, $\frac{\partial H}{\partial y} = -7$, and $\|\nabla H\|^2 = 2^2 + 7^2 = 53$ gives us the line

$$\mathbf{r}(t) = \langle 2t + 1, -7t + 3, 53t + 5000\rangle.$$

(b) **Method 1:**

The tangent plane to a function of 2 variables is the graph of the linearization:

$$L(x, y) = H(1, 3) + H_x(1, 3)(x - 1) + H_y(1, 3)(y + 7)$$

$$\nabla H(1, 3) = 2\mathbf{i} - 7\mathbf{j} \implies H_x(1, 3) = 2 \ \ and \ \ H_y(1, 3) = -7$$

$$\implies L(x, y) = 5000 + 2(x - 1) - 7(y - 3)$$

$$\implies z = 5000 + 2(x - 1) - 7(y - 3)$$

Putting this into standard form we have:

$$\boxed{2x - 7y - z = -5019}$$

Method 2:

To find the equation of a plane, we need a point and a normal vector.

To find the vector normal to the surface $z = H(x, y)$ we can rewrite it as one level surface of a function of 3 variables. i.e. let

$$g(x, y, z) = H(x, y) - z$$

(Notice that the graph of $z = H(x, y)$ is the level surface $g(x, y, z) = 0$)

We know that $\nabla g(x, y, z)$ is normal to the level surfaces of $g(x, y, z)$.
Thus

$$\nabla g(x, y, z) = \mathbf{n} = \left\langle \frac{\partial H}{\partial x}, \frac{\partial H}{\partial y}, -1 \right\rangle.$$

The equation for a plane with normal vector $\mathbf{n}$ containing the point $\mathbf{p}_0$ is

$$\mathbf{n} \cdot (r - \mathbf{p}_0) = 0.$$

Plugging in the values for this problem gives

$$2(x - 1) - 7(y - 3) - (z - 5000) = 0$$

Finally, we put this equation in standard form

$$2x - 7y - z = -5019.$$

End Of Exam