**APPM 2350—Exam 1 (Chapter 10)**
*Wednesday Feb 10th, 7:30pm-9pm 2021*

Show all your work and simplify your answers. Answers with no justification will receive no points. You are allowed one 8.5×11-in page of notes (ONE side). You must turn this in with your exam at the end. You may NOT use a calculator, smartphone, smartwatch, the Internet or any other electronic device.

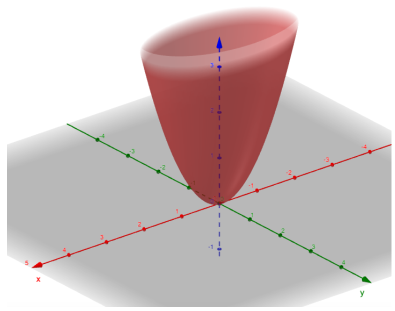
**Problem 1** (24 points)

A particle travels along the curve of intersection of the surface $z = x^2 + 3y^2$ and the plane $y = 1$.

(a) Sketch and identify (give the name of) the surface $z = x^2+3y^2$. Label any intercepts and label axes.
(b) Give a position function, $\mathbf{r}(t)$, that traces out the particle's path.
(c) What is the curvature of this curve at the point $(\sqrt{6}, 1, 9)$? Fully simplify your answer.

**SOLUTION:**

(a) It's an elliptic paraboloid, has one intercept at $(0, 0, 0)$ and looks like this:



(b) Set $y = 1$ in the equation of the paraboloid to get $z = x^2 + 3$. So if $x = t$, $z = t^2 + 3$ and the position function is

$$\boxed{\mathbf{r}(t) = \langle t, 1, t^2 + 3\rangle}$$

(c) Note that $t = \sqrt{6}$ at the point $(\sqrt{6}, 1, 9)$. We'll use the formula

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} \qquad \text{with} \qquad t = \sqrt{6}$$

To this end,

$$\mathbf{r}'(t) = \langle 1, 0, 2t\rangle \implies \mathbf{r}'(\sqrt{6}) = \langle 1, 0, 2\sqrt{6}\rangle$$

and

$$\mathbf{r}''(t) = \langle 0, 0, 2\rangle \implies \mathbf{r}''(\sqrt{6}) = \langle 0, 0, 2\rangle$$

Then

$$\mathbf{r}'(\sqrt{6}) \times \mathbf{r}''(\sqrt{6}) = \langle 1, 0, 2\sqrt{6}\rangle \times \langle 0, 0, 2\rangle = \cdots = \langle 0, -2, 0\rangle$$

so that

$$\|\mathbf{r}'(\sqrt{6}) \times \mathbf{r}''(\sqrt{6})\| = 2$$

and

$$\|\mathbf{r}'(\sqrt{6})\| = \sqrt{1^2 + 0^2 + (2\sqrt{6})^2} = 5$$

so that

$$\boxed{\kappa(\sqrt{6}) = \frac{2}{5^3} = \frac{2}{125}}$$

Note: There are other ways to calculate this curvature.

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**Problem 2** (22 points)

Gravity causes a block to roll down a (frictionless) ramp. The block follows a path that is a straight line in the $xz$-plane with slope $-\frac{1}{3}$. Suppose the block has mass $m = 0.51$ kg so that the force of gravity is $\mathbf{F} = -5\mathbf{k}$ Newtons.

(a) How much work is done by gravity if the block travels a distance of $4$ meters on the ramp?
(b) Find the force of gravity in the direction of the ramp (give your answer as a vector).

**SOLUTION:**

(a) $\mathbf{W} = \mathbf{F} \cdot \mathbf{D}$
We are given

$$\mathbf{F} = \langle 0, 0, -5 \rangle$$

Let $\mathbf{A}$ be any vector in the $xz$-plane that points downard in the direction of a line with slope $-\frac{1}{3}$. For example, let

$$\mathbf{A} = \langle 3, 0, -1 \rangle$$

Thus

$$\mathbf{D} = 4\frac{\mathbf{A}}{||\mathbf{A}||} = \frac{\langle 12, 0, -4 \rangle}{\sqrt{10}}$$

and

$$\boxed{\mathbf{W} = \langle 0, 0, -5 \rangle \cdot \frac{\langle 12, 0, -4 \rangle}{\sqrt{10}} = \frac{20}{\sqrt{10}} = 2\sqrt{10}}$$

(b) The force of gravity in the direction of the ramp is

$$proj_{\mathbf{A}}\mathbf{F} = \frac{\mathbf{A} \cdot \mathbf{F}}{||\mathbf{A}||} \frac{\mathbf{A}}{||\mathbf{A}||}$$

$$= \frac{5}{\sqrt{10}} \frac{\langle 3, 0, -1 \rangle}{\sqrt{10}}$$

$$\boxed{= \left\langle \frac{3}{2}, 0, -\frac{1}{2} \right\rangle}$$

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**Problem 3** (32 points)

Two spacecraft are planning a mid-flight hand off. Their trajectories are given by

$$\text{Spacecraft 1: } \mathbf{r_1}(t) = 8\mathbf{i} + (3t-5)\mathbf{j} + 2(t-3)^2\mathbf{k}$$

and

$$\text{Spacecraft 2: } \mathbf{r_2}(t) = 4t\mathbf{i} + (t-1)^2\mathbf{j} + 2\mathbf{k}$$

with distance measured in miles and time in minutes. To achieve this hand off, the technician will need to know the equation of the plane that contains the velocity vectors for both vessels at the time they pass each other.

(a) What are the $(x, y, z)$ coordinates of the point where the two spacecraft meet?

(b) What is the equation of the plane tangent to both of the spacecraft's trajectories at the point where they meet? (i.e. what is the equation of the plane that contains each spacecraft's tangent line at the point where they meet?) Give your answer in **standard form**, fully simplified.

(c) At the instant they meet, the first spacecraft leaves its original path, flying straight ahead along a line at a constant speed of 3 miles/min. Give the $(x, y, z)$ coordinates of Spacecraft 1 after it travels along this line for 10 minutes.

**SOLUTION:**

(a) To find the time when both spacecrafts meet we can match any of the three coordinates and then verify the others.

$$\begin{aligned} \mathbf{r}_{1x}(t) &= \mathbf{r}_{2x}(t) \\ 8 &= 4t \\ t &= 2 \end{aligned}$$

It now remains to check $t = 2$ works for all coordinates.

$$\begin{aligned} \mathbf{r}_1(2) &= 8\mathbf{i} + (3\cdot 2 - 5)\mathbf{j} + 2(2-3)^2\mathbf{k} \\ &= 8\mathbf{i} + \mathbf{j} + 2\mathbf{k} \\ \mathbf{r}_2(2) &= 4\cdot 2\mathbf{i} + (2-1)^2\mathbf{j} + 2\mathbf{k} \\ &= 8\mathbf{i} + \mathbf{j} + 2\mathbf{k} \end{aligned}$$

(b) We first need to find the tangent vector for each spacecraft.

$$\begin{aligned} \mathbf{r}_1'(t) &= 3\mathbf{j} + 4(t-3)\mathbf{k} \\ \mathbf{r}_2'(t) &= 4\mathbf{i} + 2(t-1)\mathbf{j} \end{aligned}$$

This gives us the tangent vectors $\mathbf{v}_1 = 3\mathbf{j} - 4\mathbf{k}$ and $\mathbf{v}_2 = 4\mathbf{i} + 2\mathbf{j}$. To find plane that contains the point $(8, 1, 2)$ and contains the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, the normal vector to the plane is

$$\begin{aligned} \mathbf{v}_1 \times \mathbf{v}_2 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 3 & -4 \\ 4 & 2 & 0 \end{vmatrix} \\ &= \begin{vmatrix} 3 & -4 \\ 2 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0 & -4 \\ 4 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0 & 3 \\ 4 & 2 \end{vmatrix}\mathbf{k} \\ &= 8\mathbf{i} - 16\mathbf{j} - 12\mathbf{k} \end{aligned}$$

The magnitude of normal vector is arbitrary, so we can choose $\mathbf{n} = 2\mathbf{i} - 4\mathbf{j} - 3\mathbf{k}$. The equation of the plane is now found to be

$$\begin{aligned} \mathbf{n} \cdot \langle x, y, z \rangle &= \mathbf{n} \cdot \langle 8, 1, 2 \rangle \\ 2x - 4y - 3z &= 6 \end{aligned}$$

(c) The direction that spacecraft 1 is headed at $t = 2$ is $\mathbf{d} = \dfrac{v_1}{\|v_1\|} = \dfrac{3}{5}\mathbf{j} - \dfrac{4}{5}\mathbf{k}$ and the speed is 3, and so the velocity is $V(t) = \dfrac{9}{5}\mathbf{j} - \dfrac{12}{5}\mathbf{k}$ . The position of spacecraft 1 is after 10 minutes is now found by integrating the velocity

$$\begin{aligned}
P(10) &= 8\mathbf{i} + 1\mathbf{j} + 2\mathbf{k} + \int_0^{10} V(t)\mathrm{d}t \\
&= 8\mathbf{i} + 1\mathbf{j} + 2\mathbf{k} + \left(\frac{9}{5}\mathbf{i} - \frac{12}{5}\mathbf{j}\right)t\Big|_0^{10} \\
&= 8\mathbf{i} + 1\mathbf{j} + 2\mathbf{k} + 18\mathbf{j} - 24\mathbf{k} \\
&= 8\mathbf{i} + 19\mathbf{j} - 22\mathbf{k}
\end{aligned}$$

The final coordinates are now $(8,\ 19, -22)$.

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**Problem 4** (22 points)

A few unrelated questions:

(a) Consider a particle whose position as a function of time $t$ is given by

$$\mathbf{r}(t) = (t - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j}$$

How far does the particle travel **along the parameterized path** from time $t = 0$ to time $t = 3\pi/2$? *Tip:* The identity $\sin^2(\frac{t}{2}) = \frac{1-\cos t}{2}$ may be helpful.

(b) A bike travels around a circular track with radius 16m, at a speed that increases with time, given by $||\mathbf{v}(t)|| = 3t^2$ m/s. Find the **magnitude** of the bike's acceleration at the instant when $t = 2$ seconds. (Tip: don't try to parameterize the bike's path during this exam - contact a professional later!)

<u>**SOLUTION:**</u>

**Solution:**

(a)

$$\mathbf{r}(t) = (t - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j} \quad \Rightarrow \quad \mathbf{r}'(t) = (1 - \cos t)\mathbf{i} + \sin t\mathbf{j}$$

$$\|\mathbf{r}'(t)\| = \sqrt{(1 - \cos t)^2 + (\sin t)^2} = \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t} = \sqrt{2 - 2\cos t}$$

$$= \sqrt{4\left(\frac{1 - \cos t}{2}\right)} = 2\sqrt{\sin^2\left(\frac{t}{2}\right)} = 2\left|\sin\left(\frac{t}{2}\right)\right|$$

$$\sin\left(\frac{t}{2}\right) \geq 0, \quad 0 \leq t \leq \frac{3\pi}{2} \quad \Rightarrow \quad \|\mathbf{r}'(t)\| = 2\sin\left(\frac{t}{2}\right), \quad 0 \leq t \leq \frac{3\pi}{2}$$

$$D = \int_0^{\frac{3\pi}{2}} \|\mathbf{r}'(t)\| dt = 2\int_0^{\frac{3\pi}{2}} \sin\left(\frac{t}{2}\right) dt = -4\cos\left(\frac{t}{2}\right)\Bigg|_{t=0}^{\frac{3\pi}{2}}$$

$$D = -4\left[\cos\left(\frac{3\pi}{4}\right) - \cos(0)\right] = (-4)\left(-\frac{1}{\sqrt{2}} - 1\right) = (4)\left(\frac{\sqrt{2} + 2}{2}\right)$$

$$D = \boxed{4 + 2\sqrt{2} = 2\sqrt{2}(1 + \sqrt{2})}$$

(b)

$$\|\mathbf{v}(t)\| = 3t^2 \text{ m/s}$$

$$\mathbf{a}(t) = a_T(t)\mathbf{T}(t) + a_N(t)\mathbf{N}(t)$$

$$a_T(t) = \frac{d}{dt}\left[\|\mathbf{v}(t)\|\right] = \frac{d}{dt}\left[3t^2\right] = 6t \quad \Rightarrow \quad a_T(2) = (6)(2) = 12 \text{ m/s}^2$$

$$a_N(t) = \kappa(t)\|\mathbf{v}(t)\|^2 = \left(3t^2\right)^2 \kappa(t) = 9t^4\kappa(t)$$

The (constant) radius of curvature of the circular track is $\rho = 16$ m $\quad \Rightarrow \quad \kappa = \dfrac{1}{\rho} = \dfrac{1}{16}$

$$a_N(t) = \frac{9t^4}{16} \quad \Rightarrow \quad a_N(2) = \frac{(9)(2)^4}{16} = 9 \text{ m/s}^2$$

$\mathbf{a}(2) = 12\mathbf{T}(2) + 9\mathbf{N}(2)\,,$ where $\mathbf{T}$ and $\mathbf{N}$ are orthogonal unit vectors

$$\Rightarrow \quad \|\mathbf{a}(2)\|^2 = [a_T(2)]^2 + [a_N(2)]^2 \quad \Rightarrow \quad \|\mathbf{a}(2)\| = \sqrt{(12)^2 + (9)^2} = \boxed{15 \text{ m/s}^2}$$

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End Of Exam