INSTRUCTIONS: Write your name and your instructor's name on the front of your work. Work all problems. Show your work clearly. Note that a correct answer with incorrect or no supporting work may receive no credit, while an incorrect answer with relevant work may receive partial credit.

---

1. (36 points) Given

$$g(x,y) = -\sin(\pi xy) + 3\pi xy^2$$

(a) Find the derivative of $g(x,y)$ at $(x,y) = (2,1)$ in the direction toward the *point* $(x,y) = (3,-1)$.

(b) In which direction(s) is the derivative of $g(x,y)$ at $(2,1)$ equal to 0? (Give your answer(s) as unit vector(s)).

(c) What is the *value* of the largest derivative of $g(x,y)$ at the point $(2,1)$?

Solution:

(a) The direction is defined by $\langle 3,-1\rangle - \langle 2,1\rangle = \langle 1,-2\rangle$. The unit vector defining this direction is therefore $\hat{u} = \frac{1}{\sqrt{5}}\langle 1,-2\rangle$. Thus,

$$\begin{aligned} D_{\hat{u}}g(x,y)\big|_{(x,y)=(2,1)} &= \nabla g(x,y)\big|_{(x,y)=(2,1)} \cdot \hat{u} \\ &= \langle -\pi y\cos(\pi xy) + 3\pi y^2, -\pi x\cos(\pi xy) + 6\pi xy\rangle\big|_{(x,y)=(2,1)} \cdot \frac{1}{\sqrt{5}}\langle 1,-2\rangle \\ &= \langle 2\pi, 10\pi\rangle \cdot \frac{1}{\sqrt{5}}\langle 1,-2\rangle \\ &= -\frac{18\pi}{\sqrt{5}} \end{aligned}$$

(b) The direction $\hat{v} = \langle a,b\rangle$ where the derivative is zero is defined by having $\nabla g(x,y)\big|_{(x,y)=(2,1)} \cdot \hat{v} = 0$. Thus,

$$\begin{aligned} \langle 2\pi, 10\pi\rangle \cdot \langle a,b\rangle = 2\pi(a+5b) = 0 \\ \Rightarrow a = -5b \end{aligned}$$

Any vectors following this relationship points in the direction where the derivative is zero. Letting $b = \pm 1$ gives the vectors $\langle \mp 5, \pm 1\rangle$. The corresponding unit vectors are then $\frac{1}{\sqrt{26}}\langle 5,-1\rangle$ and $\frac{1}{\sqrt{26}}\langle -5,1\rangle$.

(c) The largest value the directional derivative can take at the point $(2,1)$ is $\left|\left|\nabla g(x,y)\big|_{(x,y)=(2,1)}\right|\right| = ||\langle 2\pi, 10\pi\rangle|| = \sqrt{104}\pi = 2\pi\sqrt{26}$.

■

2. (14 points) The gravitational force on an object of mass m (located at the point (x, y, z)), due to another object of mass M (located at the origin), is given by the vector field:

$$\mathbf{F} = -GmM\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$$

where G, m, and M are constants.

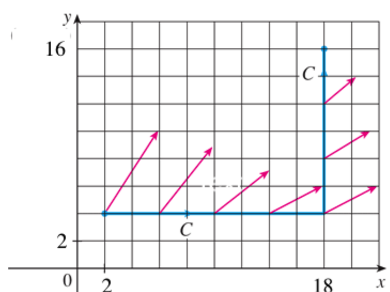
Assume an object of mass M is located at the origin. Let P_1 and P_2 be points at a distance s_1 and s_2 (respectively) from the origin. The work done by the gravitational force field $\mathbf{F}$ as an object (of mass m) moves from P_1 to P_2 is independent of the path taken. Find this work. (Please leave your final answer in terms of s_1, s_2, G, m, and M). Fully justify your answer.

Solution:
The work is given by $\int_C \vec{F} \cdot d\vec{r}$. Since this is path independent this means that $\vec{F}$ is a conservative vector field and can be written as $\vec{F} = \nabla f(x, y, z)$. The scalar potential function is then $f(x, y, z) = \frac{GmM}{\sqrt{x^2+y^2+z^2}} = \frac{GmM}{\rho}$. The work done can then be found using the Fundamental Theorem for line integrals,

$$\int_C \vec{F} \cdot d\vec{r} = f(\text{end}) - f(\text{start})$$
$$= GmM\left(\frac{1}{s_2} - \frac{1}{s_1}\right)$$

■

3. (12 points) Several vectors from the vector field $\mathbf{G}$ are shown below in pink. The path C, starts at $(x, y) = (2, 4)$ and ends at $(x, y) = (18, 16)$ as shown in blue in the figure. Use the figure below to *estimate* $\int_C \mathbf{G} \cdot \hat{\mathbf{T}}\, ds$ using the definition. (Provide the best estimate possible using *only the information given in the figure*). Fully justify your answer.



Solution:
The line integral can be approximated by a sum of $\vec{G} \cdot \hat{T}\Delta s$ values along C. Since $\vec{G}$ values are given every 4 units along C we set $\Delta s = 4$. $\hat{T}$ along the horizontal portion of C is $\hat{\imath}$ while along the vertical portion is $\hat{\jmath}$. The plot then indicates that $\vec{G} \cdot \hat{T} = 4$ along the horizontal portion of C while $\vec{G} \cdot \hat{T} = 2$ along the vertical portion. Thus,

$$\int_C \vec{G} \cdot \hat{T}\, ds \approx 4(4)(4) + 3(4)(2) = 88$$

■

4. (36 points) Consider the vector field given by

$$\mathbf{F} = yz\mathbf{i} - y\mathbf{j} + z^2\mathbf{k}$$

and let S denote the open surface that consists of the portion of the paraboloid described by $y = 10 - x^2 - z^2$ for $y \geq 1$.

(a) Fully set-up (do not evaluate), integral(s) to find $\bar{y}$, the y-component of the centroid of the surface S. For full credit, set this up using the coordinate system that leads to the simplest and fewest number of integral(s).

(b) Let $I = \iint\limits_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS$, where $\hat{\mathbf{n}}$ is the unit normal to the surface S that points outward (i.e. away from the origin).

i. Fully set-up (do not evaluate), *surface integral(s)* to directly evaluate I as defined above. For full credit, set this up using the coordinate system that leads to the simplest and fewest number of integral(s) and fully simplify the integrand(s).

ii. Fully set-up (do not evaluate), a *path integral* that would also lead to the value of I. Clearly name any theorem(s) that you use here.

Solution:

(a)

$$\bar{y} = \frac{\iint\limits_S y\, dS}{\iint\limits_S dS} = \frac{\iint\limits_R y \frac{||\nabla g||}{|\nabla g \cdot \hat{\mathbf{p}}|}\, dA}{\iint\limits_R \frac{||\nabla g||}{|\nabla g \cdot \hat{\mathbf{p}}|}\, dA}$$

Let $g(x, y, z) = x^2 + z^2 + y \implies \nabla g = \langle 2x, 1, 2z \rangle$

$$\implies ||\nabla g|| = \sqrt{4x^2 + 4z^2 + 1}$$

Easiest to project onto the xz-plane, where the shadow of the surface is caused by the intersection of $y = 1$ and $y = 10 - x^2 - z^2$. IE the region of integration in the xz-plane is the circle of radius 3 centered at the origin and

$$dS = \frac{||\nabla g||}{|\nabla g \cdot \hat{\mathbf{j}}|}\, dA = \frac{\sqrt{4x^2 + 4z^2 + 1}}{1}\, dA$$

Thus

$$\bar{y} = \frac{\iint\limits_R y\sqrt{4x^2 + 4z^2 + 1}\, dA}{\iint\limits_R \sqrt{4x^2 + 4z^2 + 1}\, dA} = \frac{\iint\limits_R (10 - x^2 - z^2)\sqrt{4x^2 + 4z^2 + 1}\, dA}{\iint\limits_R \sqrt{4x^2 + 4z^2 + 1}\, dA}$$

Switch to polar. Let $x = r\cos\theta, \quad z = r\sin\theta$

$$\bar{y} = \frac{\int_0^{2\pi} \int_0^3 (10 - r^2) r\sqrt{4r^2 + 1}\, dr d\theta}{\int_0^{2\pi} \int_0^3 r\sqrt{4r^2 + 1}\, dr d\theta}$$

i.

$$\nabla \times \mathbf{F} = \langle 0, y, -z \rangle$$

We use the same g and project onto xz-as above. Thus

$$\iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS = \iint_R \langle 0, y, -z\rangle \cdot \frac{\nabla g}{|\nabla g \cdot \mathbf{j}|}\, dA$$

$$= \iint_R \langle 0, y, -z\rangle \cdot \frac{\langle 2x, 1, 2z\rangle}{|1|}\, dA$$

$$= \iint_R (y - 2z^2)\, dA$$

$$= \iint_R (10 - x^2 - z^2 - 2z^2)\, dA$$

Switch to polar. Let $x = r\cos\theta, \quad z = r\sin\theta$

$$I = \int_0^{2\pi} \int_0^3 (10 - r^2 - 2r^2 \sin^2\theta) r\, dr d\theta$$

ii. We can apply Stokes' Theorem, since $\mathbf{F}$ has continuous partial derivatives and S is an open surface with boundary C given by the circle in the plane $y = 1$ centered at $(0, 1, 0)$ with radius 3, oriented *clockwise* in the plane (since $\mathbf{n}$ points away from the origin). Thus

$$\iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS = \oint_C \mathbf{F} \cdot d\mathbf{r}$$

One possible parameterization for the circle C (note, we want the clockwise orientation in the plane $y = 1$):

$$\mathbf{r}(t) = \langle 3\cos t, 1, -3\sin t\rangle, \quad 0 \le t \le 2\pi$$

$$\mathbf{F} = yz\mathbf{i} - y\mathbf{j} + z^2\mathbf{k} \implies \mathbf{F}(\mathbf{r}(t)) = \langle -3\sin t, -1, 9\sin^2 t\rangle$$

Thus

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

$$= \int_0^{2\pi} \langle -3\sin t, -1, 9\sin^2 t\rangle \cdot \langle -3\sin t, 0, -3\cos t\rangle\, dt$$

$$= \int_0^{2\pi} 9\sin^2 t - 27\sin^2 t\cos t\, dt = \int_0^{2\pi} 9\sin^2 t(1 - 3\cos t)\, dt$$

■

CONTINUED ON THE NEXT PAGE

5. (26 points) Given the plane

$$z = 8x - y$$

(a) Find the point on the plane that is closest to the point $(9, 4, 2)$.

(b) Find every point on the surface $2x^2 - y^2 + z^2 = 2$ where the tangent plane to the surface is parallel to the plane $z = 8x - y$.

Solution:

(a) **OPTION 1- Find the intersection of the normal line with the plane**:
Find the normal line to the plane that passes through the point $(9, 4, 2)$. The plane $z = 8x - y$ has normal vector $\mathbf{n_2} = \langle 8, -1, -1 \rangle$
Normal line:

$$x = 9 + 8t, \quad y = 4 - t, \quad z = 2 - t$$

Then the intersection of this line with the plane will give us the point on the plane closest to $(9, 4, 2)$.

$$\begin{aligned} z &= 8x - y \\ \implies 2 - t &= 8(9 + 8t) - (4 - t) \\ \implies 66t + 66 &= 0 \\ \implies t &= -1 \end{aligned}$$

$$\implies x = 1, \quad y = 5, \quad z = 3$$

thus the point on the plane closest to $(9, 4, 2)$ is $(1, 5, 3)$

OPTION 2 Use a projection:
Let $\overrightarrow{OP}$ be the position vector from the origin to the point $(9, 4, 2)$. A normal vector to the plane is given by $\mathbf{n} = \langle 8, -1, -1 \rangle$.
Let $\mathbf{v}$ be a vector from a point on the plane to the point $(9, 4, 2)$. For example, $(1, 1, 7)$ is on the plane, in which case $\mathbf{v} = \langle 8, 3, -5 \rangle$
Thus, the position vector of the point on the plane that is closest to $(9, 4, 2)$ is given by

$$\overrightarrow{OP} - proj_{\mathbf{n}}\mathbf{v}$$

$$= \langle 9, 4, 2 \rangle - \frac{\langle 8, 3, -5 \rangle \cdot \langle 8, -1, -1 \rangle}{66} \langle 8, -1, -1 \rangle = \langle 9, 4, 2 \rangle - \frac{66}{66} \langle 8, -1, -1 \rangle = \langle 1, 5, 3 \rangle$$

,
where we subtract the projection vector (instead of adding it) because in this case $\mathbf{n} \cdot \mathbf{v} = 66 > 0$
Converting this position vector back to (x, y, z) coordinates we have that the point on the plane closest to $(9, 4, 2)$ is $(1, 5, 3)$

OPTION 3: Use Lagrange multipliers
Easiest to let $f(x, y, z) = (dist)^2 = (x - 9)^2 + (y - 4)^2 + (z - 2)^2$.
Let $g(x, y, z) = 8x - y - z$.
Thus we want to solve the system of equations given by $\nabla f = \lambda \nabla g$ AND $8x - y = z$.

$$\begin{aligned} 2(x - 9) = 8\lambda &\implies \lambda = \frac{1}{4}(x - 9) \\ 2(y - 4) = -\lambda &\implies \lambda = -2(y - 4) \\ 2(z - 2) = -\lambda &\implies \lambda = -2(z - 2) \end{aligned}$$

Thus we have

$$\frac{1}{4}(x - 9) = -2(y - 4) = -2(z - 2)$$

$$\implies y = z + 2 \text{ and } x = -8z + 25$$

Substituting this into the constraint $z = 8x - y$ we have $z = 8(-8z + 25) - z - 2 \implies z = 3$ Similarly, $x = -8(3) + 25 = 1$ and $y = 3 + 2 = 5$.
Thus the point on the plane that minimizes the distance is $(x, y, z) = (1, 5, 3)$

(b) The tangent plane to the surface $2x^2 - y^2 + z^2 = 2$ has normal vector $\mathbf{n_1} = \langle 4x, -2y, 2z \rangle$
The plane $z = 8x - y$ has normal vector $\mathbf{n_2} = \langle 8, -1, -1 \rangle$
For these planes to be parallel we need their normal vectors to be parallel. i.e. they need to be scalar multiples of one another. Thus we need to solve the system:

$$\mathbf{n_1} = k\mathbf{n_2} \quad \text{and } 2x^2 - y^2 + z^2 = 2$$

where k is a scalar multiple.
This leads to the following system:

$$4x = 8k \implies x = 2k \tag{1}$$

$$-2y = -k \implies y = \frac{k}{2} \tag{2}$$

$$2z = -k \implies z = -\frac{k}{2} \tag{3}$$

$$\tag{4}$$

Plugging this into the equation for the surface we have

$$2(2k)^2 - (\frac{k}{2})^2 + (-\frac{k}{2})^2 = 2$$

$$\implies k^2 = \frac{1}{4} \implies k = \pm\frac{1}{2}$$

Thus,

$$k = \frac{1}{2} \implies (x, y, z) = (1, \frac{1}{4}, -\frac{1}{4})$$

and

$$k = -\frac{1}{2} \implies (x, y, z) = (-1, -\frac{1}{4}, \frac{1}{4})$$

from which we conclude that the tangent plane is parallel to the given plane at the points on the surface given by

$$(x, y, z) = (1, \frac{1}{4}, -\frac{1}{4}) \text{ and } (x, y, z) = (-1, -\frac{1}{4}, \frac{1}{4})$$

■

6. (26 points) Let

$$\mathbf{F} = y\mathbf{i} - x\mathbf{j} + (4z^2 + 4)\mathbf{k}$$

Consider the finite object bounded on the top by the surface $z = 1$, on the bottom by $z = 0$, and on the sides by $x^2 + y^2 + z^2 = 4$.

(a) *Calculate* the outward flux of the vector field $\mathbf{F}$ over the entire surface of the object by separately calculating the outward flux through each side and adding together. Be sure to clearly identify the flux over each part of the bounding surface. **Pro Tip:** When evaluating, choose the coordinate system that leads to the simplest and fewest number of integrals.

(b) Verify your calculation in part(a) by re-doing the problem using a key theorem in Calculus III. State the theorem you use and *fully evaluate* any integral(s). **Pro Tip:** When evaluating, choose the coordinate system that leads to the simplest and fewest number of integrals.

<u>Solution:</u>

(a)

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_{top}} \mathbf{F}\cdot\mathbf{n}\,dS + \iint_{S_{bottom}} \mathbf{F}\cdot\mathbf{n}\,dS + \iint_{S_{side}} \mathbf{F}\cdot\mathbf{n}\,dS$$

$$\iint_{S_{top}} \mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_{top}} \mathbf{F}\cdot\mathbf{k}\,dA = \iint_{S_{top}} (4z^2+4)\,dA = \iint_{S_{top}} 8\,dA = 8(\text{area of circle of radius } \sqrt{3}) = 24\pi$$

$$\iint_{S_{bottom}} \mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_{bottom}} \mathbf{F}\cdot -\mathbf{k}\,dA = \iint_{S_{bottom}} -(4z^2+4)\,dA = \iint_{S_{top}} -4\,dA = -4(\text{area of circle of radius } 2) = -16\pi$$

$$\iint_{S_{side}} \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R \mathbf{F}\cdot\frac{\nabla g}{|\nabla g\cdot\mathbf{p}|}\,dA$$

Where $g(x,y,z) = x^2+y^2+z^2 \implies \nabla g = \langle 2x, 2y, 2z\rangle$

It's easiest to let $\mathbf{p} = \mathbf{k}$ and project onto the xy-plane, in which case R is the circular annulus with inner radius $\sqrt{3}$ and outer radius 2.

$$\begin{aligned}
\iint_R \mathbf{F}\cdot\frac{\nabla g}{|\nabla g\cdot\mathbf{p}|}\,dA &= \iint_R \langle y, -x, 4z^2+4\rangle\cdot\frac{\langle 2x,2y,2z\rangle}{2|z|}\,dA \\
&= \iint_R (4z^2+4)\,dA \\
&= \iint_R (16-4x^2-4y^2+4)\,dA \\
&= \int_0^{2\pi}\int_{\sqrt{3}}^2 (20-4r^2)r\,dr\,d\theta \\
&= \left(\int_0^{2\pi} d\theta\right)\left(\int_{\sqrt{3}}^2 (20-4r^2)r\,dr\right) = 2\pi\left(10r^2 - r^4)\Big|_{\sqrt{3}}^2\right) = 2\pi(3) = 6\pi
\end{aligned}$$

Thus the total outward flux is

$$24\pi - 16\pi + 6\pi = 14\pi$$

(b) Since $\mathbf{F}$ has continuous partial derivatives and S is a closed surface, we can also calculate the outward flux across S using the Divergence Theorem:

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_D \nabla\cdot\mathbf{F}\,dV$$

$$\nabla\cdot\mathbf{F} = 8z$$

Thus we want to evaluate

$$\iiint_D 8z\,dV$$

Easiest to switch to cylindrical coordinates with the order $dr\,dz\,d\theta$ (if you use either spherical or cylindrical with the ordering $dz\,dr\,d\theta$ you will need 2 triple integrals)

$$\iiint_D 8z\,dV = \int_0^{2\pi}\int_0^1\int_0^{\sqrt{4-z^2}} 8zr\,dr\,dz\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 4z(4-z^2)\,dz\,d\theta = \left(\int_0^{2\pi} d\theta\right)\left(4\int_0^1 (4z-z^3)\,dz\right) = 8\pi\left(2z^2 - \frac{z^4}{4}\Big|_0^1\right) = 8\pi(\frac{7}{4}) = 14\pi$$

■

ENJOY YOUR SUMMER AND STAY SAFE!