1. [40 pts] Consider the surface, $S$, given by $z = 1 - x^2$, $x \geq 0$, $z \geq 0$ and $-2 \leq y \leq 2$, with the orientation of its boundary, $C$, as shown in the figure.

(a) Find the value of $\int_C xy \, dx + yz \, dy + xz \, dz$ by evaluating an appropriate surface integral using the integration order $dy \, dz$.

(b) An infinite number of dragonflies have been buzzing around in closed paths in the presence of the force

$$F = (y + yz) \mathbf{i} + (x + 3z^3 + xz) \mathbf{j} + (9yz^2 + xy - 1) \mathbf{k}$$

Each has a “work-o-meter” that measures the amount of work done by the force on each trip, and all of the meters of all the dragonflies measure zero after every one of the closed-path trips. Rest assured, they have tested all of the closed paths with the same result: 0 work. The surface, $S$, is now immersed in this vector field and our friendly scorpion scurries along the surface $S$, following the path $\mathbf{r}(t) = \sin^3(\pi t/2) \mathbf{i} + (4t - 2) \mathbf{j} + \left[1 - \sin^6(\pi t/2)\right] \mathbf{k}$, with $0 \leq t \leq 1$. How much work is done by the force on the scorpion as it travels along this path?

SOLUTION:

(a) Note that the integral can be written as

$$\int_C xy \, dx + yz \, dy + xz \, dz = \int_C \mathbf{F} \cdot d\mathbf{r} \quad \text{where} \quad \mathbf{F} = xy \mathbf{i} + yz \mathbf{j} + xz \mathbf{k}$$

so Stokes’ Theorem yields

$$\int_C xy \, dx + yz \, dy + xz \, dz = \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$$

with the surface, $S$, being the shaded region in the figure with normal pointing downward for proper orientation, given the orientation of $C$. The problem statement tells us that we must project onto the $yz$–plane, implying that $\mathbf{p} = \mathbf{i}$ with the region of integration, $\mathcal{R}$, given as $-2 \leq y \leq 2$ and $0 \leq z \leq 1$. We have

$$g(x, y, z) = z + x^2 \implies \nabla g = (2x, 0, 1) \quad \text{and} \quad |\nabla g \cdot \mathbf{p}| = |2x| = 2x \quad \text{since} \quad x \geq 0$$

Also,

$$\nabla \times \mathbf{F} = \left| \begin{array}{ccc} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial / \partial x & \partial / \partial y & \partial / \partial z \\ xy & yz & xz \end{array} \right| = -y \mathbf{i} - z \mathbf{j} - x \mathbf{k}$$

So, using $-\nabla g$ for the correct normal to the surface (given the orientation of the boundary curve),

$$\nabla \times \mathbf{F} \cdot \frac{-\nabla g}{|\nabla g \cdot \mathbf{p}|} = \left(\mathbf{i}, -z, -x\right) \cdot \frac{(-2x, 0, -1)}{2x} = \frac{2xy + x}{2x} = y + \frac{1}{2}$$

and

$$\int_C xy \, dx + yz \, dy + xz \, dz = \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathcal{R}} \nabla \times \mathbf{F} \cdot \frac{-\nabla g}{|\nabla g \cdot \mathbf{p}|} \, dA$$

$$= \int_0^1 \int_{-2}^2 \left(y + \frac{1}{2}\right) \, dy \, dz = \int_0^1 \left(\frac{y^2}{2} + \frac{y}{2}\right)_{-2}^2 \, dz = \int_0^1 2 \, dz = 2$$
2. In your blue book, write the word **TRUE** or **FALSE** as appropriate. No partial credit will be given. No justification is required.

(a) The function \( g(x, y) = (2x^2 + 3y^2)e^{-x^2-y^2} \) attains a minimum value on the triangle with vertices \((0, 0), (2, 0), \) and \((0, 2)\).

(b) Let \( \mathbf{F}(x, y, z) = P(x, y, z) \mathbf{i} + Q(x, y, z) \mathbf{j} + R(x, y, z) \mathbf{k} \) where \( P, Q, R \) all have continuous second order partial derivatives. Then \( \nabla \cdot (\nabla \times \mathbf{F}) = \nabla \times (\nabla \cdot \mathbf{F}) \).

(c) The equation \( x^2 + 6x - y^2 - 2y + z^2 = -8 \) is the equation of a hyperboloid of two sheets.

(d) \( \lim_{(x, y) \to (0, 0)} \frac{5xy}{x^2+y^2} = 0 \)

(e) Suppose \( f(a, b) = c, f_x(a, b) = f_y(a, b) = 0, \) and \( f_{xx}(a, b)f_{yy}(a, b) - f_{xy}^2(a, b) = 10. \) Then the tangent plane and first order Taylor polynomial for \( f(x, y) \) at \((a, b)\) can both be written as \( z = c. \)

(f) The path of a particle is given by \( \mathbf{r}(t) = 2 \sin t \mathbf{i} + 10 \mathbf{j} + 2 \cos t \mathbf{k} \) for \( t \geq 0. \) The unit binormal, \( \mathbf{B}(t), \) to the particle’s path is \( \mathbf{j}. \)

(g) Consider the CU campus as part of an \( xy\)-plane. You are wandering around campus on a smooth curve in this \( xy\)-plane. You have both of your arms stretched straight out from your sides. The temperature at each point on campus is a function of \( x \) and \( y \) only. At a particular instant in time as you are walking along, the rate of change of temperature with respect to time vanishes, and at this time your left hand is colder than your right hand. True or False: at that particular point in time the temperature gradient vector points in the direction of your left hand.

(h) Suppose that the \( k\)-component of the curl of a certain vector velocity field \( \mathbf{V}(x, y) \) is a nonzero constant, \( a. \) Then the circulation of \( \mathbf{V} \) along a positively oriented, smooth, simple, closed curve \( C \) in the \( xy\)-plane equals \( a \) times the area of the region, \( D, \) enclosed by \( C. \)
3. (20 pts) Your enjoyment in Calculus 3 is given by the function \( E(x, y) = x^2 + y^2 \), where \( x \) is the time you spend in your instructor’s office hours and \( y \) is the time you spend sleeping (and perhaps dreaming about Calculus 3). If the time in office hours and sleeping are such that \( x^2 - 2x + y^2 - 4y = 0 \), find your maximum enjoyment and how much time you spend in office hours and sleeping to attain this enjoyment.

**Solution:**

The function we are maximizing is \( E(x, y) = x^2 + y^2 \) and the constraint is \( g(x, y) = x^2 - 2x + y^2 - 4y = 0 \). We use Lagrange multipliers with \( E_x = 2x \), \( E_y = 2y \), \( g_x = 2x - 2 \), \( g_y = 2y - 4 \), leading to the following system of equations.

\[
\begin{align*}
2x &= \lambda(2x - 2) \implies x = \lambda(x - 1) \quad (1) \\
2y &= \lambda(2y - 4) \implies y = \lambda(y - 2) \quad (2) \\
x^2 - 2x + y^2 - 4y &= 0 \quad (3)
\end{align*}
\]

\( x \neq 1 \) in Eq. (1) and \( y \neq 2 \) in Eq. (2) so we can write

\[
\frac{x}{x - 1} = \frac{y}{y - 2} \implies xy - 2x = xy - y \implies y = 2x
\]
Using this in Eq. (3) yields
\[ x^2 - 2x + (2x)^2 - 4(2x) = 5x^2 - 10x = 5x(x - 2) = 0 \implies x = 0, 2 \implies y = 0, 4 \]
so that \((x, y) = (0, 0)\) and \((x, y) = (2, 4)\) are critical points. Now \(E(0, 0) = 0\) and \(E(2, 4) = 2^2 + 4^2 = 20\) so the maximum enjoyment is 20 which occurs when 2 time units are spent in office hours and 4 units of time are spent sleeping.

4. [20 pts] Let \(S\) be a thin lamina consisting of the portion of \(2z = \sqrt{1 + x^2 + y^2}\) lying inside the cylinder of radius 2 aligned along the \(z\)-axis. If the density of the material that comprises \(S\) is \(\delta(x, y, z) = \frac{z(x + 5)}{\sqrt{5x^2 + 5y^2 + 4}}\) g/cm\(^2\), find the mass of \(S\).

**SOLUTION:**
To find the mass, we compute the integral of the density over the surface, \(S\), which is the top branch of a hyperboloid of two sheets. We project the surface onto the \(xy\)-plane so that \(p = k\) and the region of integration is \(x^2 + y^2 \leq 4\) (that part of \(S\) inside the cylinder).

\[
g(x, y, z) = \sqrt{1 + x^2 + y^2} - 2z \implies \nabla g = \left(\frac{x}{1 + x^2 + y^2}, \frac{y}{1 + x^2 + y^2}, -2\right) \implies |\nabla g \cdot \mathbf{p}| = |-2| = 2
\]

Then (using the surface to eliminate \(z\))

\[
\text{Mass} = \iint_S \delta(x, y, z) \, dS = \iint_R \delta(x, y, z) |\nabla g| \, dA = \iint_{x^2+y^2\leq 4} \frac{z(x+5)}{\sqrt{5x^2+5y^2+4}} \sqrt{\frac{5x^2+5y^2+4}{1+x^2+y^2}} \left(\frac{1}{2}\right) \, dA
\]

\[
= \frac{1}{4} \int_{x^2+y^2\leq 4} (x+5) \, dx \, dy = \frac{1}{4} \int_0^{2\pi} \int_0^2 (r \cos \theta + 5) \, r \, dr \, d\theta
\]

\[
= \frac{1}{4} \left[ \left( \int_0^{2\pi} \cos \theta \, d\theta \right) \left( \int_0^2 r^2 \, dr \right) + 5 \left( \int_0^{2\pi} \, d\theta \right) \left( \int_0^2 r \, dr \right) \right] = \frac{1}{4} \left[ 0 + 5(2\pi)(2) \right] = 5\pi \text{ g}
\]

5. [30 pts] Let \(E\) be the portion of the solid ball \(x^2 + y^2 + z^2 \leq 4\) lying below the third quadrant, that is, where \(x \leq 0, y \leq 0, z \leq 0\). You are tasked with finding the outward flux of the vector field \(\mathbf{F} = x^2 \mathbf{i} - 2xy \mathbf{j} + 3xz \mathbf{k}\) across the boundary of \(E\).

(a) Write the equation(s) of the surface(s) that form the boundary of \(E\).

(b) Rather than doing surface integrals, use a Calculus 3 theorem to find this flux.

**SOLUTION:**
(a) The boundary of \(E\) is comprised of 4 surfaces: \(S_1: z = 0, S_2: y = 0, S_3: x = 0,\) and the surface of the sphere \(S_4: x^2 + y^2 + z^2 = 4\). The boundary is thus \(S = S_1 \cup S_2 \cup S_3 \cup S_4\). Finding the flux through this boundary, \(S\), requires evaluation of 4 surface integrals.

(b) Note that \(S\) is closed so that it and \(E\) are candidates for Gauss’ Divergence Theorem. Thus, with

\[
\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(-2xy) + \frac{\partial}{\partial z}(3xz) = 2x - 2x + 3x = 3x
\]

we have

\[
\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \nabla \cdot \mathbf{F} \, dV = 3 \int_{-2}^{0} \int_{-\sqrt{4-x^2}}^{0} \int_{-\sqrt{4-x^2-y^2}}^{0} x \, dz \, dy \, dx
\]

A switch to spherical coordinates will make things easier. Thus
\[ \iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = 3 \int^3_{\pi/2} \int^\pi_{\pi/2} \int_0^2 (\rho \sin \phi \cos \theta) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \]

\[ = 3 \left( \int^3_{\pi/2} \cos \theta \, d\theta \right) \left( \int^\pi_{\pi/2} \sin^2 \phi \, d\phi \right) \left( \int_0^2 \rho^3 \, d\rho \right) \]

Note: \( \sin^2 \phi = \frac{1}{2} - \frac{\cos 2\phi}{2} \)

\[ = 3 \left( \sin \theta \bigg|_\pi^{3\pi/2} \right) \left( \frac{\phi}{2} - \frac{1}{4} \sin 2\phi \right) \left| \rho^4 \right|_{\pi/2}^\pi \]

\[ = 3 (-1 - 0) \left[ (\frac{\pi}{2} - 0) - (\frac{\pi}{4} - 0) \right] \left( \frac{2^4}{4} - 0 \right) \]

\[ = 3 (-1) \left( \frac{\pi}{4} \right) (4) = -3\pi \]