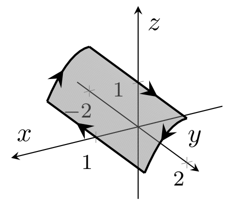
1. [40 pts] Consider the surface, $\mathcal{S}$, given by $z = 1 - x^2,\ x \geq 0,\ z \geq 0$ and $-2 \leq y \leq 2$, with the orientation of its boundary, $\mathcal{C}$, as shown in the figure.



(a) Find the value of $\int_{\mathcal{C}} xy\,\mathrm{d}x + yz\,\mathrm{d}y + xz\,\mathrm{d}z$ by evaluating an appropriate surface integral using the integration order $\mathrm{d}y\,\mathrm{d}z$.

(b) An infinite number of dragonflies have been buzzing around in closed paths in the presence of the force

$$\mathbf{F} = (y + yz)\,\mathbf{i} + (x + 3z^3 + xz)\,\mathbf{j} + (9yz^2 + xy - 1)\,\mathbf{k}$$

Each has a "work-o-meter" that measures the amount of work done by the force on each trip, and all of the meters of all the dragonflies measure zero after every one of the closed-path trips. Rest assured, they have tested all of the closed paths with the same result: 0 work. The surface, $\mathcal{S}$, is now immersed in this vector field and our friendly scorpion scurries along the surface $\mathcal{S}$, following the path $\mathbf{r}(t) = \sin^3(\pi t/2)\,\mathbf{i} + (4t-2)\,\mathbf{j} + \left[1 - \sin^6(\pi t/2)\right]\mathbf{k}$, with $0 \leq t \leq 1$. How much work is done by the force on the scorpion as it travels along this path?

SOLUTION:

(a) Note that the integral can be written as

$$\int_{\mathcal{C}} xy\,\mathrm{d}x + yz\,\mathrm{d}y + xz\,\mathrm{d}z = \int_{\mathcal{C}} \mathbf{F}\cdot\mathrm{d}\mathbf{r} \quad \text{where} \quad \mathbf{F} = xy\,\mathbf{i} + yz\,\mathbf{j} + xz\,\mathbf{k}$$

so Stokes' Theorem yields

$$\int_{\mathcal{C}} xy\,\mathrm{d}x + yz\,\mathrm{d}y + xz\,\mathrm{d}z = \int_{\mathcal{C}} \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \iint_{\mathcal{S}} \boldsymbol{\nabla}\times\mathbf{F}\boldsymbol{\cdot}\mathrm{d}\mathbf{S}$$

with the surface, $\mathcal{S}$, being the shaded region in the figure with normal pointing downward for proper orientation, given the orientation of $\mathcal{C}$. The problem statement tells us that we must project onto the yz–plane, implying that $\mathbf{p} = \mathbf{i}$ with the region of integration, $\mathcal{R}$, given as $-2 \leq y \leq 2$ and $0 \leq z \leq 1$. We have

$$g(x,y,z) = z + x^2 \implies \nabla g = \langle 2x, 0, 1\rangle \text{ and } |\nabla g\cdot\mathbf{p}| = |2x| = 2x \ \text{ since } x \geq 0$$

Also,

$$\boldsymbol{\nabla}\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & yz & xz \end{vmatrix} = -y\,\mathbf{i} - z\,\mathbf{j} - x\,\mathbf{k}$$

So, using $-\nabla g$ for the correct normal to the surface (given the orientation of the boundary curve),

$$\boldsymbol{\nabla}\times\mathbf{F}\boldsymbol{\cdot}\frac{-\nabla g}{|\nabla g\cdot\mathbf{p}|} = \langle -y, -z, -x\rangle\cdot\frac{\langle -2x, 0, -1\rangle}{2x} = \frac{2xy + x}{2x} = y + \frac{1}{2}$$

and

$$\int_{\mathcal{C}} xy\,\mathrm{d}x + yz\,\mathrm{d}y + xz\,\mathrm{d}z = \iint_{\mathcal{S}} \boldsymbol{\nabla}\times\mathbf{F}\boldsymbol{\cdot}\mathrm{d}\mathbf{S} = \iint_{\mathcal{R}} \boldsymbol{\nabla}\times\mathbf{F}\boldsymbol{\cdot}\frac{-\nabla g}{|\nabla g\cdot\mathbf{p}|}\,\mathrm{d}A$$

$$= \int_0^1\int_{-2}^2 \left(y + \frac{1}{2}\right)\mathrm{d}y\,\mathrm{d}z = \int_0^1 \left(\frac{y^2}{2} + \frac{y}{2}\right)\bigg|_{-2}^{2}\mathrm{d}z = \int_0^1 2\,\mathrm{d}z = 2$$

(b) The vector field is conservative so we need only find the potential function, f, for $\mathbf{F}$ and evaluate it at the endpoints of the path, $\mathbf{r}(0) = \langle 0, -2, 1\rangle$ and $\mathbf{r}(1) = \langle 1, 2, 0\rangle$.

$$\frac{\partial f}{\partial x} = y + yz \implies f(x,y,z) = \int (y+yz)\,\mathrm{d}x = xy(1+z) + g(y,z)$$

$$\frac{\partial f}{\partial y} = x(1+z) + \frac{\partial g}{\partial y} = x + 3z^3 + xz \implies \frac{\partial g}{\partial y} = 3z^3 \implies g(y,z) = \int 3z^3\,\mathrm{d}y \implies g(y,z) = 3yz^3 + h(z)$$

$$\implies f(x,y,z) = xy(1+z) + 3yz^3 + h(z)$$

$$\frac{\partial f}{\partial z} = xy + 9yz^2 + \frac{\mathrm{d}h}{\mathrm{d}z} = 9yz^2 + xy - 1 \implies \frac{\mathrm{d}h}{\mathrm{d}z} = -1 \implies h(z) = -z + C$$

$$\implies f(x,y,z) = xy(1+z) + 3yz^3 - z + C$$

So the work done by the force on the scorpion is

$$\text{Work} = \int_{\mathcal{C}} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_{\mathcal{C}} \nabla f \cdot \mathrm{d}\mathbf{r} = f(\mathbf{r}(1)) - f(\mathbf{r}(0)) = f(1,2,0) - f(0,-2,1)$$
$$= (1)(2)(1+0) + 3(-2)(0^3) - 0 + C - [(0)(-2)(1+1) + (3)(-2)(1^3) - 1 + C] = 9$$

Although not the recommended approach, since the vector field is conservative, line integrals are path independent so we can compute the actual line integral along any path, for example, the straight line segment between $\langle 0,-2,1\rangle$ and $\langle 1,2,0\rangle$. This can be parameterized as

$$\mathbf{r}(t) = (1-t)\langle 0,-2,1\rangle + t\langle 1,2,0\rangle = \langle t, -2+4t, 1-t\rangle, \quad 0 \le t \le 1$$
$$\implies \mathbf{F}(\mathbf{r}(t)) = \langle -4t^2 + 10t - 4, -3t^3 + 8t^2 - 7t + 3, 36t^3 - 86t^2 + 70t - 19\rangle$$
$$\implies \mathbf{r}'(t) = \langle 1, 4, -1\rangle$$
$$\implies \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t) = -48t^3 + 114t^2 - 88t + 27$$

so that

$$\text{Work} = \int_0^1 \left(-48t^3 + 114t^2 - 88t + 27\right)\mathrm{d}t = \left(-12t^4 + \frac{114}{3}t^3 - 44t^2 + 27t\right)\Bigg|_0^1 = 9$$

■

2. [40 pts] In your blue book, write the word **TRUE** or **FALSE** as appropriate. No partial credit will be given. No justification is required.

(a) The function $g(x,y) = (2x^2 + 3y^3)e^{-x^2-y^2}$ attains a minimum value on the triangle with vertices $(0,0)$, $(2,0)$, and $(0,2)$.

(b) Let $\mathbf{F}(x,y,z) = P(x,y,z)\,\mathbf{i} + Q(x,y,z)\,\mathbf{j} + R(x,y,z)\,\mathbf{k}$ where P, Q, R all have continuous second order partial derivatives. Then $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{F}) = \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\cdot\mathbf{F})$.

(c) The equation $x^2 + 6x - y^2 - 2y + z^2 = -8$ is the equation of a hyperboloid of two sheets.

(d) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{5xy}{x^2+y^2} = 0$

(e) Suppose $f(a,b) = c$, $f_x(a,b) = f_y(a,b) = 0$, and $f_{xx}(a,b)f_{yy}(a,b) - f_{xy}^2(a,b) = 10$. Then the tangent plane and first order Taylor polynomial for $f(x,y)$ at (a,b) can both be written as $z = c$.

(f) The path of a particle is given by $\mathbf{r}(t) = 2\sin t\,\mathbf{i} + 10\,\mathbf{j} + 2\cos t\,\mathbf{k}$ for $t \ge 0$. The unit binormal, $\mathbf{B}(t)$, to the particle's path is $\mathbf{j}$.

(g) Consider the CU campus as part of an xy–plane. You are wandering around campus on a smooth curve in this xy–plane. You have both of your arms stretched straight out from your sides. The temperature at each point on campus is a function of x and y only. At a particular instant in time as you are walking along, the rate of change of temperature with respect to time vanishes, and at this time your left hand is colder than your right hand. True or False: at that particular point in time the temperature gradient vector points in the direction of your left hand.

(h) Suppose that the **k**–component of the curl of a certain vector velocity field $\mathbf{V}(x,y)$ is a nonzero constant, a. Then the circulation of $\mathbf{V}$ along a positively oriented, smooth, simple, closed curve $\mathcal{C}$ in the xy–plane equals a times the area of the region, $\mathcal{D}$, enclosed by $\mathcal{C}$.

SOLUTION:

(a) **TRUE**: $g(x,y)$ is continuous and the triangle is a closed, bounded region. The Extreme Value Theorem then implies that $g(x,y)$ will attain a minimum (and a maximum!).

(b) **FALSE**: $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \cdot \mathbf{F})$ is meaningless (cannot take the curl of a scalar)

(c) **FALSE**: It is a cone. Complete the square :

$$x^2 + 6x + 9 - 9 - \left(y^2 + 2y + 1 - 1\right) + z^2 = -8 \implies (x+3)^2 - (y+1)^2 + z^2 = 0$$

(d) **FALSE**: Consider the path $y = mx$. Then

$$\lim_{(x,y)\to(0,0)} \frac{5xy}{x^2+y^2} = \lim_{(x,mx)\to(0,0)} \frac{5x(mx)}{x^2+(mx)^2} = \frac{5m}{1+m^2}$$

implying that the limit depends on the path of approach to the origin, further implying that the limit does not exist.

(e) **TRUE**: $f(a,b)$ is a local extremum so the tangent plane is horizontal there $(z = c)$. The tangent plane and the first order Taylor polynomial are the same thing.

(f) **TRUE**:

$$\mathbf{r}(t) = \langle 2\sin t, 10, 2\cos t\rangle \implies \mathbf{r}'(t) = \langle 2\cos t, 0, -2\sin t\rangle \implies \|\mathbf{r}'(t)\| = 2$$
$$\implies \mathbf{T}(t) = \langle \cos t, 0, -\sin t\rangle \implies \mathbf{T}'(t) = \langle -\sin t, 0, -\cos t\rangle \implies \|\mathbf{T}'(t)\| = 1$$
$$\implies \mathbf{N}(t) = \langle -\sin t, 0, -\cos t\rangle$$
$$\implies \mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & 0 & -\sin t \\ -\sin t & 0 & -\cos t \end{vmatrix} = \mathbf{j}$$

(g) **FALSE**: Your hands being different temperatures implies that the temperature gradient vector is not the zero vector. The fact that you are walking implies that your velocity is nonzero. Thus,

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \nabla T \cdot \mathrm{r}'(t) = 0 \implies \nabla T \perp \mathbf{r}'(t)$$

and your path is tangent to the level curve of $T(x,y)$. The temperature gradient vector lies along your arms and points in the direction of maximum increase of T, pointing in the direction of your right hand (from cold to warm).

(h) **TRUE**: If $\mathbf{k} \cdot (\boldsymbol{\nabla} \times \mathbf{V}) = a$ then Green's Theorem yields

$$\oint_{\mathcal{C}} \mathbf{V} \cdot \mathrm{d}\mathbf{r} = \iint_{\mathcal{D}} \mathbf{k} \cdot (\boldsymbol{\nabla} \times \mathbf{V})\, \mathrm{d}A = \iint_{\mathcal{D}} a\, \mathrm{d}A = a \times \text{area}(\mathcal{D})$$

■

3. [20 pts] Your enjoyment in Calculus 3 is given by the function $E(x,y) = x^2 + y^2$, where x is the time you spend in your instructor's office hours and y is the time you spend sleeping (and perhaps dreaming about Calculus 3). If the time in office hours and sleeping are such that $x^2 - 2x + y^2 - 4y = 0$, find your maximum enjoyment and how much time you spend in office hours and sleeping to attain this enjoyment.

SOLUTION:

The function we are maximizing is $E(x,y)$ and the constraint is $g(x,y) = x^2 - 2x + y^2 - 4y = 0$. We use Lagrange multipliers with $E_x = 2x,\;\; E_y = 2y,\;\; g_x = 2x - 2,\;\; g_y = 2y - 4$, leading to the following system of equations.

$$2x = \lambda(2x-2) \implies x = \lambda(x-1) \tag{1}$$
$$2y = \lambda(2y-4) \implies y = \lambda(y-2) \tag{2}$$
$$x^2 - 2x + y^2 - 4y = 0 \tag{3}$$

$x \neq 1$ in Eq. (1) and $y \neq 2$ in Eq. (2) so we can write

$$\frac{x}{x-1} = \lambda = \frac{y}{y-2} \implies xy - 2x = xy - y \implies y = 2x$$

Using this in Eq. (3) yields

$$x^2 - 2x + (2x)^2 - 4(2x) = 5x^2 - 10x = 5x(x-2) = 0 \implies x = 0, 2 \implies y = 0, 4$$

so that $(x, y) = (0, 0)$ and $(x, y) = (2, 4)$ are critical points. Now $E(0, 0) = 0$ and $E(2, 4) = 2^2 + 4^2 = 20$ so the maximum enjoyment is 20 which occurs when 2 time units are spent in office hours and 4 units of time are spent sleeping. ■

4. [20 pts] Let $\mathcal{S}$ be a thin lamina consisting of the portion of $2z = \sqrt{1 + x^2 + y^2}$ lying inside the cylinder of radius 2 aligned along the z–axis. If the density of the material that comprises $\mathcal{S}$ is $\delta(x, y, z) = \dfrac{z(x+5)}{\sqrt{5x^2 + 5y^2 + 4}}$ g/cm^2, find the mass of $\mathcal{S}$.

SOLUTION:

To find the mass, we compute the integral of the density over the surface, $\mathcal{S}$, which is the top branch of a hyperboloid of two sheets. We project the surface onto the xy–plane so that $\mathbf{p} = \mathbf{k}$ and the region of integration is $x^2 + y^2 \leq 4$ (that part of $\mathcal{S}$ inside the cylinder).

$$g(x, y, z) = \sqrt{1 + x^2 + y^2} - 2z \implies \nabla g = \left\langle \frac{x}{\sqrt{1 + x^2 + y^2}}, \frac{y}{\sqrt{1 + x^2 + y^2}}, -2 \right\rangle \implies |\nabla g \cdot \mathbf{p}| = |-2| = 2$$

$$\implies \|\nabla g\| = \sqrt{\frac{x^2}{1 + x^2 + y^2} + \frac{y^2}{1 + x^2 + y^2} + 4} = \sqrt{\frac{5x^2 + 5y^2 + 4}{1 + x^2 + y^2}}$$

Then (using the surface to eliminate z)

$$\text{Mass} = \iint_{\mathcal{S}} \delta(x, y, z)\, \mathrm{d}S = \iint_{\mathcal{R}} \delta(x, y, z) \frac{\|\nabla g\|}{|\nabla g \cdot \mathbf{p}|}\, \mathrm{d}A = \iint\limits_{x^2+y^2 \leq 4} \frac{z(x+5)}{\sqrt{5x^2 + 5y^2 + 4}} \sqrt{\frac{5x^2 + 5y^2 + 4}{1 + x^2 + y^2}} \left(\frac{1}{2}\right) \mathrm{d}A$$

$$= \frac{1}{4} \iint\limits_{x^2+y^2 \leq 4} (x + 5)\, \mathrm{d}x\, \mathrm{d}y = \frac{1}{4} \int_0^{2\pi} \int_0^2 (r\cos\theta + 5)\, r\, \mathrm{d}r\, \mathrm{d}\theta$$

$$= \frac{1}{4} \left[\left(\int_0^{2\pi} \cos\theta\, \mathrm{d}\theta \right) \left(\int_0^2 r^2\, \mathrm{d}r \right) + 5 \left(\int_0^{2\pi} \mathrm{d}\theta \right) \left(\int_0^2 r\, \mathrm{d}r \right) \right] = \frac{1}{4} \left[0 + 5(2\pi)(2) \right] = 5\pi \text{ g}$$

■

5. [30 pts] Let $\mathcal{E}$ be the portion of the solid ball $x^2 + y^2 + z^2 \leq 4$ lying below the third quadrant, that is, where $x \leq 0$, $y \leq 0$, $z \leq 0$. You are tasked with finding the outward flux of the vector field $\mathbf{F} = x^2\, \mathbf{i} - 2xy\, \mathbf{j} + 3xz\, \mathbf{k}$ across the boundary of $\mathcal{E}$.

   (a) Write the equation(s) of the surface(s) that form the boundary of $\mathcal{E}$.

   (b) Rather than doing surface integrals, use a Calculus 3 theorem to find this flux.

SOLUTION:

   (a) The boundary of $\mathcal{E}$ is comprised of 4 surfaces: $\mathcal{S}_1$: $z = 0$, $\mathcal{S}_2$: $y = 0$, $\mathcal{S}_3$: $x = 0$, and the surface of the sphere $\mathcal{S}_4$: $x^2 + y^2 + z^2 = 4$. The boundary is thus $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_4$. Finding the flux through this boundary, $\mathcal{S}$, requires evaluation of 4 surface integrals.

   (b) Note that $\mathcal{S}$ is closed so that it and $\mathcal{E}$ are candidates for Gauss' Divergence Theorem. Thus, with

$$\boldsymbol{\nabla} \cdot \mathbf{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(-2xy) + \frac{\partial}{\partial z}(3xz) = 2x - 2x + 3x = 3x$$

   we have

$$\iint_{\mathcal{S}} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iiint\limits_{\mathcal{E}} \boldsymbol{\nabla} \cdot \mathbf{F}\, \mathrm{d}V = 3 \int_{-2}^0 \int_{-\sqrt{4-x^2}}^0 \int_{-\sqrt{4-x^2-y^2}}^0 x\, \mathrm{d}z\, \mathrm{d}y\, \mathrm{d}x$$

   A switch to spherical coordinates will make things easier. Thus

$$\begin{aligned}
\iint_{\mathcal{S}} \mathbf{F} \cdot \mathrm{d}\mathbf{S} &= 3 \int_{\pi}^{3\pi/2} \int_{\pi/2}^{\pi} \int_{0}^{2} (\rho \sin\phi \cos\theta) \rho^2 \sin\phi \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}\theta \\
&= 3 \left(\int_{\pi}^{3\pi/2} \cos\theta \, \mathrm{d}\theta \right) \left(\int_{\pi/2}^{\pi} \sin^2\phi \, \mathrm{d}\phi \right) \left(\int_{0}^{2} \rho^3 \, \mathrm{d}\rho \right) \quad \text{Note: } \left(\sin^2\phi = \frac{1}{2} - \frac{\cos 2\phi}{2} \right) \\
&= 3 \left(\sin\theta \Big|_{\pi}^{3\pi/2} \right) \left(\frac{\phi}{2} - \frac{1}{4} \sin 2\phi \right) \Bigg|_{\pi/2}^{\pi} \frac{\rho^4}{4} \Bigg|_{0}^{2} \\
&= 3 \, (-1 - 0) \left[\left(\frac{\pi}{2} - 0 \right) - \left(\frac{\pi}{4} - 0 \right) \right] \left(\frac{2^4}{4} - 0 \right) \\
&= 3(-1) \left(\frac{\pi}{4} \right) (4) = -3\pi
\end{aligned}$$

■