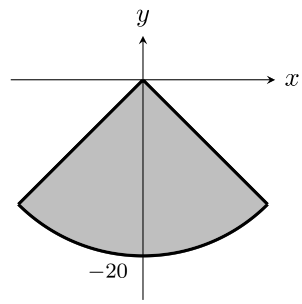
1. [20 pts] Sam the spider studier has been in the field taking measurements and has found that the density of scorpions in a certain area can be described by the function $\delta(x, y) = 100e^{-(x/2)^2-(y/2)^2}$ scorpions per square meter. You are buying a house in this area which will be centered at the origin of a Cartesian coordinate system, with the positive y–axis pointing north. Your yard will be a quarter of a disk of radius 20 meters, centered on your house and located to the south, bounded by the lines $y = \pm x$.
   (a) [5 pts] Sketch the region depicting your yard.
   (b) [15 pts] How many scorpions can you expect to have in your yard? (Your final answer need not be an integer)

SOLUTION:

(a) Sketch of your yard.



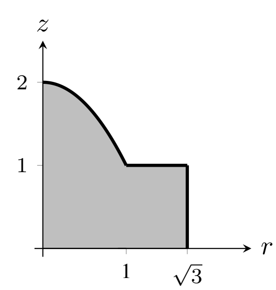
(b) The shape of the yard and the form of the density function strongly suggest the use of polar coordinates. To that end, the region is given by $0 \le r \le 20$ and $5\pi/4 \le \theta \le 7\pi/4$ and the density function becomes

$$100e^{-(x/2)^2-(y/2)^2} = 100e^{-(x^2+y^2)/4} = 100e^{-r^2/4}$$

Then

$$\text{Total scorpions} = \int_{5\pi/4}^{7\pi/4} \int_0^{20} 100e^{-r^2/4}\, r\, \mathrm{d}r\, \mathrm{d}\theta \overset{u=-r^2/4}{=} 100 \left(\int_{5\pi/4}^{7\pi/4} \mathrm{d}\theta \right) \left(2 \int_{-100}^{0} e^u\, \mathrm{d}u \right) = 100\pi(1 - e^{-100})$$

2. [32 pts] You are in charge of making three-dimensional concrete structures to place on the tops of the towers of a castle. A cross section of half of the structure is depicted in the following rz–plane (constant θ plane). The curved portion of the structure is $x^2 + y^2 + z = 2$. The density of the concrete is $\delta(x, y, z) = x + y + 4$.



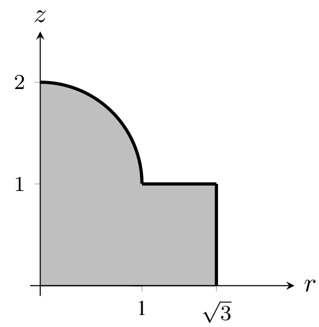
   (a) [13 pts] Find the mass of the concrete needed to build the three dimensional structure by setting up, **but not evaluating**, the appropriate integral(s) using cylindrical coordinates and the order $\mathrm{d}r\, \mathrm{d}z\, \mathrm{d}\theta$
   (b) [19 pts] Now suppose that only the curved part of the structure is replaced with a portion of the sphere of radius 1, centered at $(x, y, z) = (0, 0, 1)$, while the rectangular portion remains the same.
      i. [2 pts] Make a sketch of this new structure in the rz–plane (constant θ plane).
      ii. [17 pts] Using spherical coordinates and the order $\mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\theta$, set up, **but do not evaluate**, the appropriate integral(s) to find the mass of the concrete needed to build this new structure.

SOLUTION:

(a) The equations describing the upper portion of the structure are $z = 2 - r^2$ for $0 \le r \le 1$ and $z = 1$ for $1 \le r \le \sqrt{3}$. The density is $\delta(r, \theta, z) = r\cos\theta + r\sin\theta + 4$. Thus

$$\text{Mass} = \int_0^{2\pi}\int_1^2\int_0^{\sqrt{2-z}} (r\cos\theta + r\sin\theta + 4)\, r\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}\theta + \int_0^{2\pi}\int_0^1\int_0^{\sqrt{3}} (r\cos\theta + r\sin\theta + 4)\, r\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}\theta$$

(b) i. Sketch of the new region.



ii. The rectangular equation for the sphere is $x^2 + y^2 + (z-1)^2 = 1 \implies x^2 + y^2 + z^2 = 2z \implies \rho = 2\cos\phi$ in spherical coordinates. The horizontal line has rectangular equation $z = 1 \implies \rho = \sec\phi$ in spherical coordinates. The vertical line is $r = \sqrt{3} \implies \rho = \sqrt{3}\csc\phi$ in spherical coordinates. The density is $\delta(\rho, \phi, \theta) = \rho\sin\phi(\cos\theta + \sin\theta) + 4$ in spherical coordinates. Thus

$$\begin{aligned}\text{Mass} &= \int_0^{2\pi}\int_0^{\pi/4}\int_0^{2\cos\phi} \left[\rho\sin\phi(\cos\theta+\sin\theta)+4\right]\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta \\ &+ \int_0^{2\pi}\int_{\pi/4}^{\pi/3}\int_0^{\sec\phi} \left[\rho\sin\phi(\cos\theta+\sin\theta)+4\right]\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta \\ &+ \int_0^{2\pi}\int_{\pi/3}^{\pi/2}\int_0^{\sqrt{3}\csc\phi} \left[\rho\sin\phi(\cos\theta+\sin\theta)+4\right]\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta\end{aligned}$$

■

3. [28 pts] As part of an environmental engineering study, you are flying a drone around collecting air pollution particles in a filter. The density of the particles in the region where the drone is flying is $\delta(x, y, z) = x^2 y(z+1)$ particles per meter. Your flight path occurs in the plane $z = 3$. It begins at $(x, y, z) = (-2, 0, 3)$, follows a straight line to $(x, y, z) = (0, 2, 3)$ then follows a quarter of the circle of radius 2, centered at $(0, 0, 3)$ with $y \ge 0$ and $x \le 0$, back to the starting point.

(a) [4 pts] Sketch the drone's flight in the plane $z = 3$, making certain to show the correct direction of the flight.

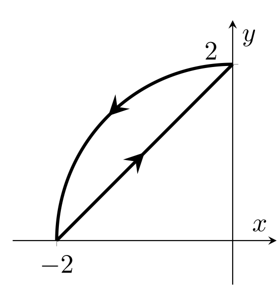
(b) [8 pts] Parameterize the path of the drone.

(c) [8 pts] Find the total number (not necessarily an integer) of particles collected in the filter during the straight line segment of the flight.

(d) [8 pts] If the wind is producing a force field described by $\mathbf{F} = \left(x^2 + y^2\right)\mathbf{i} - 2xy\,\mathbf{j} + 5z\,\mathbf{k}$, find the work done by this force during the circular portion of the flight.

SOLUTION:

(a) Sketch of drone's flight.



(b) Line segment ($\mathcal{C}_\ell$):

$$\mathbf{r}_\ell(t) = (1-t)\langle -2,0,3\rangle + t\langle 0,2,3\rangle = \langle -2+2t, 2t, 3\rangle, \quad \mathbf{r}'_\ell(t) = \langle 2,2,0\rangle, \quad 0 \le t \le 1$$

Portion of circle ($\mathcal{C}_c$):

$$\mathbf{r}_c(t) = \langle 2\cos t, 2\sin t, 3\rangle, \quad \mathbf{r}'_c(t) = \langle -2\sin t, 2\cos t, 0\rangle, \quad \pi/2 \le t \le \pi$$

(c)

$$\text{Number of particles} = \int_{\mathcal{C}_\ell} x^2 y(z+1)\,\mathrm{d}s = \int_0^1 (2t-2)^2(2t)(3+1)\|\langle 2,2,0\rangle\|\mathrm{d}t$$

$$= 8\sqrt{8}\int_0^1 t\left(4t^2 - 8t + 4\right)\mathrm{d}t = 64\sqrt{2}\int_0^1 \left(t^3 - 2t^2 + t\right)\mathrm{d}t$$

$$= 64\sqrt{2}\left(\frac{1}{4}t^4 - \frac{2}{3}t^3 + \frac{1}{2}t^2\right)\Bigg|_0^1 = 64\sqrt{2}\left(\frac{3}{12} - \frac{8}{12} + \frac{6}{12}\right) = \frac{16\sqrt{2}}{3}$$

(d) $\mathbf{F}(\mathbf{r}(t)) = 4\,\mathbf{i} - 2(2\cos t)(2\sin t)\,\mathbf{j} + 15\,\mathbf{k} = 4\,\mathbf{i} - 8\cos t\sin t\,\mathbf{j} + 15\,\mathbf{k} \implies \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t) = -8\sin t - 16\cos^2 t\sin t$

$$\text{Work} = \int_{\mathcal{C}_c} \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_{\pi/2}^{\pi}\left(-8\sin t - 16\cos^2 t\sin t\right)\mathrm{d}t$$

$$= 8\cos t\Big|_{\pi/2}^{\pi} + 16\int_0^{-1} u^2\,\mathrm{d}u = -8 + \frac{16}{3}u^3\Bigg|_0^{-1} = -\frac{40}{3}$$

■

4. [20 pts] To impress your friends, you are designing a new living space that will be really neat. The floor of the living space will be bounded by the first quadrant portion of the curves $xy = 1$, $xy = 2$, $x^2y = 2$ and $x^2y = 4$ and its roof will have the shape of $z = x^2y\cos\left(\frac{\pi}{6}xy\right)$. Before you can get the required building permit, the city requires you to provide them with the average height of the roof. Using an appropriate change of variables from xy–space to uv–space, find the number you will give to the city.

SOLUTION:

The floor can be described by $1 \le xy \le 2$ and $2 \le x^2y \le 4$, suggesting the transformation $u = xy$ and $v = x^2y$, yielding the new region of integration as $1 \le u \le 2$ and $2 \le v \le 4$. Inverting these equations gives

$$u = xy \implies y = u/x \quad \text{and} \quad v = x^2y \implies y = v/x^2 \implies u/x = v/x^2 \implies x = v/u = u^{-1}v \quad \text{and} \quad y = u^2/v = u^2v^{-1}$$

The Jacobian of the transformation is thus

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -u^{-2}v & u^{-1} \\ 2uv^{-1} & -u^2v^{-2} \end{vmatrix} = v^{-1} - 2v^{-1} = -\frac{1}{v}$$

and we have $x^2y\cos\left(\frac{\pi}{6}xy\right) = v\cos\frac{\pi}{6}u$ so that

$$\text{Average height} = \frac{\text{Volume under roof above floor}}{\text{Area of floor}} = \frac{\displaystyle\int_1^2\int_2^4 v\cos\left(\frac{\pi u}{6}\right)\left|-\frac{1}{v}\right|\mathrm{d}v\,\mathrm{d}u}{\displaystyle\int_1^2\int_2^4\left|-\frac{1}{v}\right|\mathrm{d}v\,\mathrm{d}u} = \frac{\displaystyle\int_1^2\int_2^4 \cos\frac{\pi u}{6}\,\mathrm{d}v\,\mathrm{d}u}{\displaystyle\int_1^2\int_2^4 \frac{1}{v}\,\mathrm{d}v\,\mathrm{d}u}$$

$$= \frac{\left(\displaystyle\int_1^2 \cos\frac{\pi u}{6}\,\mathrm{d}u\right)\left(\displaystyle\int_2^4 \mathrm{d}v\right)}{\left(\displaystyle\int_1^2 \mathrm{d}u\right)\left(\displaystyle\int_2^4 \frac{1}{v}\,\mathrm{d}v\right)} = \frac{\left(\frac{6}{\pi}\sin\frac{\pi u}{6}\Big|_1^2\right)\left(v\Big|_2^4\right)}{\left(u\Big|_1^2\right)\left(\ln|v|\Big|_2^4\right)} = \frac{\frac{6}{\pi}(\sin\frac{\pi}{3} - \sin\frac{\pi}{6})(2)}{\ln 4 - \ln 2} = \frac{6(\sqrt{3}-1)}{\pi\ln 2}$$

Note: If $u = x^2y$ and $v = xy$, then $x = u/v$, $y = v^2/u$ and $J(u,v) = 1/u$. ■