Write on the front of your bluebook a grading key, your name, student ID, your lecture number and instructor. This exam is worth 150 points and has 4 questions.

- Submit this exam sheet with your bluebook. However, nothing on this exam sheet will be graded. Make sure all of your work is in your bluebook.
- **Show all work and simplify your answers!** Answers with no justification will receive no points unless otherwise noted. **Please begin each problem on a new page.**
- You are taking this exam in a proctored and honor code enforced environment. Thus, no notes/papers, calculators, cell phones, or other electronic devices are permitted.

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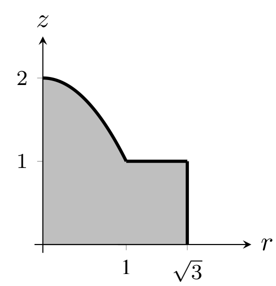
1. [20 pts] Sam the spider studier has been in the field taking measurements and has found that the density of scorpions in a certain area can be described by the function $\delta(x,y) = 100e^{-(x/2)^2-(y/2)^2}$ scorpions per square meter. You are buying a house in this area which will be centered at the origin of a Cartesian coordinate system, with the positive y–axis pointing north. Your yard will be a quarter of a disk of radius 20 meters, centered on your house and located to the south, bounded by the lines $y = \pm x$.
   (a) [5 pts] Sketch the region depicting your yard.
   (b) [15 pts] How many scorpions can you expect to have in your yard? (Your final answer need not be an integer)

2. [32 pts] You are in charge of making three-dimensional concrete structures to place on the tops of the towers of a castle. A cross section of half of the structure is depicted in the following rz–plane (constant θ plane). The curved portion of the structure is $x^2 + y^2 + z = 2$. The density of the concrete is $\delta(x, y, z) = x + y + 4$.



   (a) [13 pts] Find the mass of the concrete needed to build the three dimensional structure by setting up, **but not evaluating**, the appropriate integral(s) using cylindrical coordinates and the order $\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}\theta$
   (b) [19 pts] Now suppose that only the curved part of the structure is replaced with a portion of the sphere of radius 1, centered at $(x, y, z) = (0, 0, 1)$, while the rectangular portion remains the same.
      i. [2 pts] Make a sketch of this new structure in the rz–plane (constant θ plane).
      ii. [17 pts] Using spherical coordinates and the order $\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$, set up, **but do not evaluate**, the appropriate integral(s) to find the mass of the concrete needed to build this new structure.

3. [28 pts] As part of an environmental engineering study, you are flying a drone around collecting air pollution particles in a filter. The density of the particles in the region where the drone is flying is $\delta(x, y, z) = x^2y(z + 1)$ particles per meter. Your flight path occurs in the plane $z = 3$. It begins at $(x, y, z) = (-2, 0, 3)$, follows a straight line to $(x, y, z) = (0, 2, 3)$ then follows a quarter of the circle of radius 2, centered at $(0, 0, 3)$ with $y \geq 0$ and $x \leq 0$, back to the starting point.
   (a) [4 pts] Sketch the drone's flight in the plane $z = 3$, making certain to show the correct direction of the flight.
   (b) [8 pts] Parameterize the path of the drone.
   (c) [8 pts] Find the total number (not necessarily an integer) of particles collected in the filter during the straight line segment of the flight.
   (d) [8 pts] If the wind is producing a force field described by $\mathbf{F} = \left(x^2 + y^2\right)\mathbf{i} - 2xy\,\mathbf{j} + 5z\,\mathbf{k}$, find the work done by this force during the circular portion of the flight.

CONTINUED ON BACK

4. [20 pts] To impress your friends, you are designing a new living space that will be really neat. The floor of the living space will be bounded by the first quadrant portion of the curves $xy = 1$, $xy = 2$, $x^2y = 2$ and $x^2y = 4$ and its roof will have the shape of $z = x^2y\cos\left(\frac{\pi}{6}xy\right)$. Before you can get the required building permit, the city requires you to provide them with the average height of the roof. Using an appropriate change of variables from xy–space to uv–space, find the number you will give to the city.

PROJECTIONS, DISTANCES FROM POINT S TO LINE CONTAINING POINT P, AND S TO PLANE WITH NORMAL $\mathbf{n}$

$$\operatorname{proj}_{\mathbf{a}}\mathbf{b} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a} \qquad d = \frac{\left\|\overrightarrow{PS}\times\mathbf{v}\right\|}{\|\mathbf{v}\|} \qquad d = \left|\overrightarrow{PS}\cdot\frac{\mathbf{n}}{\|\mathbf{n}\|}\right|$$

ARC LENGTH, FRENET FORMULAS, AND TANGENTIAL AND NORMAL ACCELERATION COMPONENTS

$$\mathrm{d}s = \|\mathbf{v}\|\,\mathrm{d}t \qquad \mathbf{T} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = \frac{\mathbf{v}}{\|\mathbf{v}\|} \qquad \mathbf{N} = \frac{\mathrm{d}\mathbf{T}/\mathrm{d}s}{\|\mathrm{d}\mathbf{T}/\mathrm{d}s\|} = \frac{\mathrm{d}\mathbf{T}/\mathrm{d}t}{\|\mathrm{d}\mathbf{T}/\mathrm{d}t\|} \qquad \mathbf{B} = \mathbf{T}\times\mathbf{N}$$

$$\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} = \kappa\mathbf{N} \qquad \frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s} = -\tau\mathbf{N} \qquad \kappa = \left\|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\right\| = \frac{\|\mathbf{v}\times\mathbf{a}\|}{\|\mathbf{v}\|^3} = \frac{|f''(x)|}{\left\{1+[f'(x)]^2\right\}^{3/2}} = \frac{|\dot{x}\ddot{y}-\dot{y}\ddot{x}|}{(\dot{x}^2+\dot{y}^2)^{3/2}} \qquad \tau = -\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s}\cdot\mathbf{N}$$

$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N} \qquad a_T = \frac{\mathrm{d}\|\mathbf{v}\|}{\mathrm{d}t} \qquad a_N = \kappa\|\mathbf{v}\|^2 = \sqrt{\|\mathbf{a}\|^2 - a_T^2}$$

DIRECTIONAL DERIVATIVE, DISCRIMINANT, AND LAGRANGE MULTIPLIERS

$$\frac{\mathrm{d}f}{\mathrm{d}s} = (\nabla f)\cdot\mathbf{u} \qquad D = f_{xx}f_{yy} - (f_{xy})^2 \qquad \nabla f = \lambda\nabla g, \quad g = 0$$

TAYLOR'S FORMULA [at the point (x_0, y_0)]

$$f(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + \frac{1}{2!}\left[f_{xx}(x_0,y_0)(x-x_0)^2 + 2f_{xy}(x_0,y_0)(x-x_0)(y-y_0) + f_{yy}(x_0,y_0)(y-y_0)^2\right]$$
$$+\frac{1}{3!}\left[f_{xxx}(x_0,y_0)(x-x_0)^3 + 3f_{xxy}(x_0,y_0)(x-x_0)^2(y-y_0) + 3f_{xyy}(x_0,y_0)(x-x_0)(y-y_0)^2 + f_{yyy}(x_0,y_0)(y-y_0)^3\right] + \cdots$$

ERROR IN LINEAR APPROXIMATION $\quad |E(x,y)| \le \frac{1}{2!}M\left(|x-a|+|y-b|\right)^2$, where $\max\{|f_{xx}|, |f_{xy}|, |f_{yy}|\} \le M$

ERROR IN QUADRATIC APPROXIMATION $\quad |E(x,y)| \le \frac{1}{3!}M\left(|x-x_0|+|y-y_0|\right)^3$, where $\max\{|f_{xxx}|, |f_{xxy}|, |f_{xyy}|, |f_{yyy}|\} \le M$

CHANGE OF VARIABLES/SUBSTITUTIONS IN MULTIPLE INTEGRALS

$$\iint_{\mathcal{R}} f(x,y)\,\mathrm{d}A = \iint_{\mathcal{S}} f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\mathrm{d}u\,\mathrm{d}v \quad \text{where} \quad J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

POLAR COORDINATES $\quad x = r\cos\theta \qquad y = r\sin\theta \qquad r^2 = x^2+y^2 \qquad \mathrm{d}A = \mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$

Coordinate Conversions

Cylindrical to Rectangular	Spherical to Rectangular	Spherical to Cylindrical
$x = r\cos\theta$	$x = \rho\sin\phi\cos\theta$	$r = \rho\sin\phi$
$y = r\sin\theta$	$y = \rho\sin\phi\sin\theta$	$z = \rho\cos\phi$
$z = z$	$z = \rho\cos\phi$	$\theta = \theta$

$$\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z = \rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$$

MASS, MOMENTS, AND CENTER OF MASS $\quad$ Mass $M = \iint_R \delta\,\mathrm{d}A \quad$ Moments $M_x = \iint_R y\,\delta\,\mathrm{d}A \quad M_y = \iint_R x\,\delta\,\mathrm{d}A \quad$ Center of mass $\overline{x} = \frac{M_y}{M} \quad \overline{y} = \frac{M_x}{M}$

FLOW AND FLUX $\quad$ Flow $= \int_C \mathbf{F}\cdot\mathbf{T}\,\mathrm{d}s = \int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_C P\,\mathrm{d}x + Q\,\mathrm{d}y \qquad$ Flux $= \int_C \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}s = \int_C P\,\mathrm{d}y - Q\,\mathrm{d}x \qquad \mathbf{n} = \mathbf{T}\times\mathbf{k}$