1. [25 pts] Our scorpion from the last exam is crawling around on the shelf again. The temperature at a point (x, y) on the shelf is $T(x, y) = x^2 + 3y^2 + 12y$.

   (a) [9 pts] Use the Second Derivatives Test to determine whether or not the scorpion can find any coldest or warmest points on the shelf away from the edge. If so, give the locations and temperatures of these points. If not, explain why not.

   (b) [16 pts] The scorpion now decides to go on an adventure, following a circular path of radius 5 with center at the origin of a Cartesian coordinate system. Use Lagrange multipliers to determine the highest and lowest temperatures, and their locations, encountered by the scorpion as it traverses this circular path.

   SOLUTION:

   (a) We need to find and classify any critical points of the temperature function.

$$T_x = 2x = 0 \implies x = 0$$

$$T_y = 6y + 12 = 0 \implies y = -2$$

   so the critical point is $(0, -2)$. We use the Second Derivatives Test to classify the critical point

$$T_{xx} = 2 \quad T_{xy} = 0 \quad T_{yy} = 6 \implies D(x, y) = T_{xx}T_{yy} - T_{xy}^2 = (2)(6) - 0^2 = 12 > 0 \text{ for all } (x, y)$$

   which, when noting that $T_{xx} > 0$ [also for all (x, y)] implies that $f(0, -2) = 0^2 + 3(-2)^2 + 12(-2) = -12$ is a local minimum. The scorpion will encounter the coldest part of the shelf at $(0, -2)$ where the temperature is -12.

   (b) The objective function (the function whose extrema we seek) is $T(x, y) = x^2 + 3y^2 + 12y$ with the constraint being the scorpion's path, $g(x, y) = x^2 + y^2 = 25$. We need to solve the following system of simultaneous equations

$$T_x = \lambda g_x \implies 2x = \lambda(2x) \implies x = \lambda x \tag{1}$$

$$T_y = \lambda g_y \implies 6y + 12 = \lambda(2y) \implies 3y + 6 = \lambda y \tag{2}$$

$$x^2 + y^2 = 25 \tag{3}$$

   Eq. (1) is equivalent to $x(1 - \lambda) = 0$ implying that $x = 0$ or $\lambda = 1$. If $x = 0$, Eq. (3) yields $y^2 = 25 \implies y = \pm 5$ so $(0, 5)$ and $(0, -5)$ are critical points. If $\lambda = 1$, Eq. (2) yields $3y + 6 = y \implies y = -3$ and Eq. (3) becomes $x^2 + (-3)^2 = 25 \implies x^2 = 16 \implies x = \pm 4$ giving $(4, -3)$ and $(-4, -3)$ as critical points. We now find the temperature at each of these critical points

$$T(0, 5) = 0^2 + 3(5)^2 + 12(5) = 135 \quad \text{(maximum)}$$

$$T(0, -5) = 0^2 + 3(-5)^2 + 12(-5) = 15$$

$$T(4, -3) = 4^2 + 3(-3)^2 + 12(-3) = 7 \quad \text{(minimum)}$$

$$T(-4, -3) = (-4)^2 + 3(-3)^2 + 12(-3) = 7 \quad \text{(minimum)}$$

■

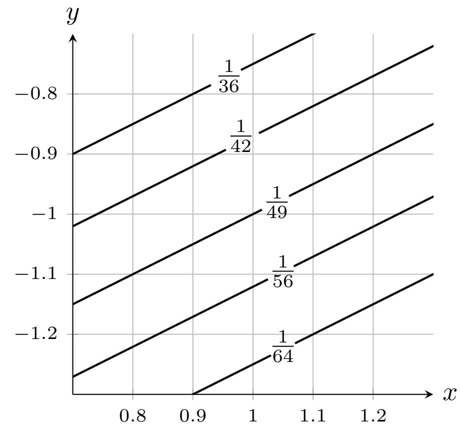
2. [20 pts] Consider the function $f(x, y) = \ln(c + 2x - 4y)$ where $|c| \leq 1$ is a constant.

   (a) [4 pts] Is f continuous at $\left(\frac{1}{2}, \frac{1}{2}\right)$? Justify your answer.

   (b) [5 pts] The graph of $f(x, y)$ is a surface. Find the curve of intersection of this surface with the xy–plane. Write your answer in the form $y = g(x)$.

   (c) [5 pts] Find the first order/linear Taylor approximation for $f(x, y)$ centered at $(1, -1)$.

   (d) [6 pts] Find an upper bound on the error in the approximation you found in part (c) if $|x - 1| \leq 0.1$ and $|y + 1| \leq 0.2$. The accompanying figure showing a set of level curves of $1/(c + 2x - 4y)^2$ is provided to help with this problem.



SOLUTION:

(a) No. Because $|c| \le 1$, $c + 2\left(\frac{1}{2}\right) - 4\left(\frac{1}{2}\right) = c - 1 \le 0$ implying that $\left(\frac{1}{2}, \frac{1}{2}\right)$ is not in the domain of f (can't take $\ln$ of a negative number), implying that $f\left(\frac{1}{2}, \frac{1}{2}\right)$ is not defined, implying that the function is not continuous there.

(b) This amounts to finding the level curve $f(x, y) = 0$.

$$\ln(c + 2x - 4y) = 0 \implies e^{\ln(c+2x-4y)} = e^0 \implies c + 2x - 4y = 1 \implies y = \frac{1}{2}x + \frac{1}{4}(c - 1)$$

(c)

$$\begin{aligned}
f(x, y) &= \ln(c + 2x - 4y) \implies f(1, -1) = \ln(c + 6) \\
f_x(x, y) &= \frac{2}{c + 2x - 4y} \implies f_x(1, -1) = \frac{2}{c + 6} \\
f_y(x, y) &= \frac{-4}{c + 2x - 4y} \implies f_y(1, -1) = \frac{-4}{c + 6} \\
T_1(x, y) &= \ln(c + 6) + \frac{2}{c + 6}(x - 1) - \frac{4}{c + 6}(y + 1)
\end{aligned}$$

(d) To use the error formula, we need to find an upper bound, M, on the absolute value of the second derivatives of $f(x, y)$ over the region of interest, $|x - 1| \le 0.1$ and $|y + 1| \le 0.2$.

$$f_{xx}(x, y) = \frac{-4}{(c + 2x - 4y)^2}, \qquad f_{xy}(x, y) = \frac{8}{(c + 2x - 4y)^2}, \qquad f_{yy}(x, y) = \frac{-16}{(c + 2x - 4y)^2}$$

From the level curves in the figure, $\dfrac{1}{(c + 2x - 4y)^2} \le \dfrac{1}{36}$ for $|x - 1| \le 0.1$ and $|y + 1| \le 0.2$. Thus

$$|f_{xx}| = \frac{4}{(c + 2x - 4y)^2} \le \frac{4}{36} = \frac{1}{9}, \qquad |f_{xy}| = \frac{8}{(c + 2x - 4y)^2} \le \frac{8}{36} = \frac{2}{9}, \qquad |f_{yy}| = \frac{16}{(c + 2x - 4y)^2} \le \frac{16}{36} = \frac{4}{9}$$

so we choose $M = \frac{4}{9}$ and

$$|E(x, y)| \le \frac{\frac{4}{9}}{2}\left((0.1 + 0.2)\right)^2 = \frac{2}{9}\left(\frac{9}{100}\right) = \frac{1}{50}$$

3. [20 pts] Cone P has a radius of 3 and a height of 4 and cone Q has a radius of 4 and height of 3. The lateral (side) surface area of a cone is given by the formula $S = \pi r\sqrt{h^2 + r^2}$. Suppose the error in measuring the cones' radius is twice the error in measuring their heights. Use differentials to estimate which cone's surface area will be more sensitive to these errors.

SOLUTION:

$$\begin{aligned}
\mathrm{d}S &= S_r\,\mathrm{d}r + S_h\,\mathrm{d}h \\
&= \pi\left(\frac{r^2}{\sqrt{h^2 + r^2}} + \sqrt{h^2 + r^2}\right)\mathrm{d}r + \frac{\pi r h}{\sqrt{h^2 + r^2}}\,\mathrm{d}h \\
&= \pi\left(\frac{2r^2 + h^2}{\sqrt{h^2 + r^2}}\right)\mathrm{d}r + \frac{\pi r h}{\sqrt{h^2 + r^2}}\,\mathrm{d}h \\
&= \frac{\pi}{\sqrt{h^2 + r^2}}\left(4r^2 + 2h^2 + rh\right)\mathrm{d}h \qquad (\mathrm{d}r = 2\,\mathrm{d}h)
\end{aligned}$$

Cone P:

$$\mathrm{d}S = \frac{\pi}{\sqrt{4^2+3^2}}\left[4(3^2)+2(4^2)+(3)(4)\right]\mathrm{d}h = \frac{80\pi}{5}\,\mathrm{d}h$$

Cone Q:

$$\mathrm{d}S = \frac{\pi}{\sqrt{3^2+4^2}}\left[4(4^2)+2(3^2)+(4)(3)\right]\mathrm{d}h = \frac{94\pi}{5}\,\mathrm{d}h$$

Thus the surface area of cone Q is more sensitive to measurement errors. ■

4. [35 pts] At a point (x, y, z) in space, the intensity of light emanating from a bug zapper is $I(x, y, z) = z^4 e^{-x^2-y^2}$. A moth is flying along the path $\mathbf{r}(t) = \sin(\pi t)\,\mathbf{i} + \cos(\pi t)\,\mathbf{j} + t^2\,\mathbf{k}$.

   (a) When the moth arrives at the point $(0, -1, 1)$:

   i. [6 pts] What is the rate of change of the light's intensity with respect to distance $(\mathrm{d}I/\mathrm{d}s)$ in the direction of $\mathbf{i} - \mathbf{j} + \mathbf{k}$?

   ii. [6 pts] What is the rate of change of the light's intensity with respect to time?

   iii. [10 pts] The moth suddenly realizes that the light is a bug zapper. In order to survive, it must change course and skedaddle in a direction that makes the light's intensity decrease at the largest rate possible. Find the direction in which the moth must begin to fly to make this happen, and determine the corresponding rate of change of the light's intensity.

   iv. [4 pts] Is there a direction in which the moth can begin flying, not necessarily on its current path, where the instantaneous rate of change of the light's intensity is $\sqrt{5}$? If so, find it. If not, explain why not.

   (b) [9 pts] After eluding the bug zapper, the moth lands on a roof whose height $z(x, y)$ above the ground is given by $z = 20 - 2x^2 - 4y^2$. Enjoying its newly found safety, the moth happily crawls around on the roof, following a path given by $x = uv$ and $y = u/v$. Using the chain rule, find the rate of change of its height above ground with respect to v when $u = 1$ and $v = 2$.

SOLUTION:

(a) The moth arrives at the point $(0, -1, 1)$ when $t = 1$ and we will need the gradient vector there to answer these questions.

$$\nabla I(x,y,z) = \left\langle -2xz^4e^{-x^2-y^2}, -2yz^4e^{-x^2-y^2}, 4z^3e^{-x^2-y^2}\right\rangle = 2z^3e^{-x^2-y^2}\langle -xz, -yz, 2\rangle$$

$$\nabla I(0,-1,1) = 2e^{-1}\langle 0,1,2\rangle$$

i. We need to compute the directional derivative. A unit vector in the direction of interest is $\frac{1}{\sqrt{3}}\langle 1,-1,1\rangle$ so

$$\left.\frac{\mathrm{d}I}{\mathrm{d}s}\right|_{(0,-1,1)} = D_{\mathbf{u}}I\,(0,-1,1) = \nabla I\,(0,-1,1)\cdot\frac{1}{\sqrt{3}}\langle 1,-1,1\rangle = 2e^{-1}\langle 0,1,2\rangle\cdot\frac{1}{\sqrt{3}}\langle 1,-1,1\rangle = \frac{2e^{-1}}{\sqrt{3}} = \frac{2\sqrt{3}}{3e}$$

ii. We'll need $\mathbf{r}'(t) = \langle \pi\cos(\pi t), -\pi\sin(\pi t), 2t\rangle$ and

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\partial I}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial I}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial I}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} = \nabla I\cdot\mathbf{r}'(t)$$

$$\left.\frac{\mathrm{d}I}{\mathrm{d}t}\right|_{t=1} = \nabla I(0,-1,1)\cdot\mathbf{r}'(1) = 2e^{-1}\langle 0,1,2\rangle\cdot\langle -\pi,0,2\rangle = \frac{8}{e}$$

iii. The moth needs to fly in the direction opposite to the gradient vector at that point with a rate of change equal to the magnitude of the gradient vector there.

$$\text{Direction:}\quad -\nabla I(0,-1,1) = -2e^{-1}\langle 0,1,2\rangle$$

$$\text{Rate:}\quad \left\|-2e^{-1}\langle 0,1,2\rangle\right\| = \frac{2\sqrt{5}}{e}$$

iv. No. The maximum rate of change of the light's intensity at this point is $\|\nabla I(0,-1,1)\| = \dfrac{2\sqrt{5}}{e} < \sqrt{5}$.

(b)

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = (-4x)(u) + (-8y)\left(-\frac{u}{v^2}\right) = -4u^2v + 8\frac{u^2}{v^3} = -4(1^2)(2) + 8\frac{1^2}{2^3} = -7$$

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