1. [20 pts] Darth Vader is flying a TIE fighter along the path given by $\mathbf{r}(t) = \cos(\pi t) \mathbf{i} + \sin(\pi t) \mathbf{j} + (4t - t^2) \mathbf{k}$, $0 \leq t \leq 4$. During the flight, he fires a laser beam from the front of his ship that travels in a straight line. When $t = 3$, find the coordinates of the point where the laser beam penetrates the $xy$-plane.

**Solution:**

The laser beam essentially follows the tangent line. To find the equation of the tangent line, we need a point on the tangent line and its direction. When $t = 3$, the fighter is at $\mathbf{r}(3) = \langle \cos(3\pi), \sin(3\pi), 4(3) - 3^2 \rangle = \langle -1, 0, 3 \rangle$, so this is a point on the curve and on the tangent line. The velocity vector at this point is $\mathbf{v}(3) = \langle -\pi \sin(3\pi), \pi \cos(3\pi), 4 - 2(3) \rangle = \langle 0, -\pi, -2 \rangle$. This gives the direction of the tangent line, so its equation is

$$\mathbf{L}(s) = \langle -1, 0, 3 \rangle + s\langle -\pi, -\pi, -2 \rangle = \langle -1 - \pi s, 0, 3 - 2s \rangle, \quad -\infty < s < \infty$$

This line intersects the $xy$-plane when its $z$-coordinate vanishes, which occurs if $3 - 2s = 0 \implies s = \frac{3}{2}$. The laser intersects the $xy$-plane at the point $(-1, -\frac{3\pi}{2}, 0)$.

2. [15 pts] Consider the equation $-\frac{1}{4}x^2 - 8y^2 + x^2 + 2z - 20 = 0$

(a) [6 pts] Name the surface, providing justification for your answer.

(b) [6 pts] Does the surface intersect the $yz$-plane? Justify your answer.

(c) [3 pts] Name the conic section of the trace when $x = -\sqrt{\frac{32}{7}}$, providing justification for your answer.

**Solution:**

(a) Complete the square.

$$x^2 - 8y^2 - \frac{1}{4}(z^2 - 8z + 16 - 16) - 20 = 0$$

$$x^2 - 8y^2 - \frac{1}{4}(z - 4)^2 = 20 - 4 = 16$$

$$-\frac{x^2}{16} + \frac{y^2}{\frac{1}{4}} + \frac{(z - 4)^2}{64} = -1$$

This is a hyperboloid of two sheets.

(b) Intersecting the $yz$-plane means that $x = 0$. Substituting this into the equation gives the result

$$\frac{y^2}{2} + \frac{(z - 4)^2}{64} = \left( \frac{y}{\sqrt{2}} \right)^2 + \left( \frac{z - 4}{8} \right)^2 = -1,$$

an equation with no solution (sum of two squares is never negative). Thus, the surface does not intersect the $yz$-plane.

(c) To find the trace, substitute $x = -\sqrt{\frac{32}{7}}$ into the equation. The result is

$$-\frac{(\sqrt{\frac{32}{7}})^2}{16} + \frac{y^2}{2} + \frac{(z - 4)^2}{64} = -1 \implies \left( \frac{y}{\sqrt{2}} \right)^2 + \left( \frac{z - 4}{8} \right)^2 = 1$$

which is an ellipse.

3. [20 pts] Find the equation of the plane that is perpendicular to the plane $2z = 5x + 4y$ and contains the line with symmetric equations $-x = \frac{y + 2}{5} = \frac{z - 5}{-4}$. Write your final answer in the form $ax + by + cz = d$.

**Solution:**

A vector $\mathbf{v}_1$ in the plane we seek, and therefore parallel to it, is the direction vector of the given line, namely $\mathbf{v}_1 = -\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}$. Since we want the plane we seek to be perpendicular to the given plane, the given plane’s normal vector will be parallel to our plane. Call this $\mathbf{v}_2 = 5\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$. With $\mathbf{v}_1$ and $\mathbf{v}_2$ parallel to our plane, their cross product will provide a normal, $\mathbf{n}$, to our plane.

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 5 & -4 \\ 5 & 4 & -2 \end{vmatrix} = 6\mathbf{i} - 22\mathbf{j} - 29\mathbf{k}$$

We need a point in the plane, which will also be on the line. To find such a point, we can rewrite the symmetric equations of the line in parametric form as

$$x = -t \quad y = -2 + 5t \quad z = 5 - 4t$$
and evaluate these when \( t = 0 \), giving a point in the sought-after plane as \((0, -2, 5)\). With a normal vector and a point, the equation of the plane is

\[
6(x - 0) - 22(y + 2) - 29(z - 5) = 0 \implies 6x - 22y - 44 - 29z + 145 = 0 \implies 6x - 22y - 29z = -101
\]

4. [45 pts] A scorpion is crawling on a shelf located 2 meters above the floor in a room. Its path is given by \( \mathbf{r}(t) = \frac{t^3}{3} \mathbf{i} + \frac{t^2}{2} \mathbf{j} + 2 \mathbf{k}, t > 0 \), with distances measured in meters. Answer ALL of the following questions for \( t = 1 \) second.

(a) [5 pts] Where is the scorpion?

(b) [5 pts] How fast is the scorpion crawling? (Include units in your answer)

(c) [10 pts] Briefly describe in words (i.e. DO NOT COMPUTE) what the following two quantities represent physically in terms of the scorpion’s path:

i. \( \sqrt{\mathbf{r}(t) \cdot \mathbf{r}(t)} \)

ii. \( \int_0^t \sqrt{\mathbf{r}'(u) \cdot \mathbf{r}'(u)} \, du \)

(d) [5 pts] Find the scorpion’s unit tangent vector, \( \mathbf{T} \).

(e) [5 pts] Find the unit normal to the scorpion’s path by computing \( \mathbf{T} \times \mathbf{k} \).

(f) [5 pts] Find the binormal vector and the equation of the scorpion’s osculating plane. Then, with the origin of an \( xy \)-coordinate system representing the scorpion’s location, point your eyes in the direction of \( \mathbf{B} \) and make a sketch of and label \( \mathbf{T} \) and \( \mathbf{N} \). You do not need to draw the actual path or location of the scorpion.

(g) [5 pts] How fast is the scorpion’s speed changing? (Include units in your answer)

(h) [5 pts] Does the scorpion’s acceleration possess a component normal to its path? If so, find it, including units in your answer. If not, explain why not.

**Solution:**

(a)

\[
\mathbf{r}(1) = \frac{1}{3} \mathbf{i} + \frac{1}{2} \mathbf{j} + 2 \mathbf{k} \text{ or } \left( \frac{1}{3}, \frac{1}{2}, 2 \right)
\]

(b) Need to find the speed, \( ||\mathbf{v}(1)|| \). We have

\[
\mathbf{v}(t) = \mathbf{r}'(t) = t^2 \mathbf{i} + tj \implies \mathbf{v}(1) = \mathbf{i} + \mathbf{j} \implies ||\mathbf{v}(1)|| = \sqrt{2} \text{ m/s}
\]

(c) i. \( \sqrt{\mathbf{r}(1) \cdot \mathbf{r}(1)} = ||\mathbf{r}(1)|| \), which is the distance the scorpion is from the origin after one second.

ii. \( \int_0^1 \sqrt{\mathbf{r}'(u) \cdot \mathbf{r}'(u)} \, du = \int_0^1 ||\mathbf{r}'(u)|| \, du \), which gives the distance the scorpion has actually crawled during the first second, that is, the arc length of the scorpion’s path over the first second.

(d) From part (b), \( ||\mathbf{r}'(t)|| = \sqrt{t^4 + t^2} = \sqrt{t^2(t^2 + 1)} = |t|\sqrt{t^2 + 1} = t\sqrt{t^2 + 1} \) since \( t > 0 \).

\[
\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||} = \frac{t^2 \mathbf{i} + t \mathbf{j}}{t\sqrt{t^2 + 1}} = \frac{t}{\sqrt{t^2 + 1}} \mathbf{i} + \frac{1}{\sqrt{t^2 + 1}} \mathbf{j} \implies \mathbf{T}(1) = \frac{1}{\sqrt{2}} \mathbf{i} + \frac{1}{\sqrt{2}} \mathbf{j}
\]

(e)

\[
\mathbf{N}(1) = \mathbf{T}(1) \times \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{1}{\sqrt{2}} \mathbf{i} - \frac{1}{\sqrt{2}} \mathbf{j}
\]

(f)

\[
\mathbf{B}(1) = \mathbf{T}(1) \times \mathbf{N}(1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{vmatrix} = -\mathbf{k}
\]

Since \( \mathbf{T} \) and \( \mathbf{N} \) have no \( k \)-component, the osculating plane is parallel to the \( xy \)-plane. Since the trajectory (path) includes the component \( 2 \mathbf{k} \), the osculating plane’s equation is \( z = 2 \).

The following figure shows \( \mathbf{T} \) and \( \mathbf{N} \). Note that \( \mathbf{B} \) is pointing into the page in the figure.
We will need the acceleration vector $a(t) = r''(t) = 2t\,i + j \implies a(1) = 2\,i + j$. The speed change is simply the tangential component of the acceleration.

$$a_T(1) = \frac{r'(1) \cdot r''(1)}{\|r'(1)\|} = \frac{(i + j) \cdot (2\,i + j)}{\sqrt{2}} = \frac{3}{\sqrt{2}} \text{ m/s}^2$$

(h) Yes.

$$a_N(1) = \frac{\|r'(1) \times r''(1)\|}{\|r'(1)\|} = \frac{\frac{1}{\sqrt{2}} \left|\begin{array}{ccc} i & j & k \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{array}\right|}{\frac{1}{\sqrt{2}}} = \frac{1}{\sqrt{2}} \| - k \| = \frac{1}{\sqrt{2}}$$

Since this is nonzero, there is an acceleration normal to the scorpion’s path.