1. [2350/102324 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

   (a) The function $g(x,y) = \begin{cases} \dfrac{\cos^2 x - \sin^2(y/2)}{\cos x - \sin(y/2)} & (x,y) \neq (0,\pi) \\ 0 & (x,y) = (0,\pi) \end{cases}$ is not continuous at $(0,\pi)$.

   (b) The linearization of $z = 2\cos x \sin y$ at the point $\left(\dfrac{\pi}{4}, \dfrac{\pi}{4}\right)$ is $L(x,y) = 1 - x + y$

   (c) The level surface of the function $w(x,y,z) = 3x^2 + y^2 + z^2$ corresponding to $w = -2$ is an ellipsoid.

   (d) If $t = u/v$ and $u = x^2 - y^2$ and $v = 4xy^3$, then $\partial t/\partial x = \left(x^2 + y^2\right)/(4x^2y^3)$.

   (e) There exists a unit vector $\mathbf{u}$ such that the rate of change of $f(x,y) = x^3 + e^y$ in the direction of $\mathbf{u}$ at $(x,y) = (1,0)$ is 10.

SOLUTION:

(a) **TRUE**

$$\lim_{(x,y)\to(0,\pi)} \frac{\cos^2 x - \sin^2(y/2)}{\cos x - \sin(y/2)} = \lim_{(x,y)\to(0,\pi)} \frac{[\cos x - \sin(y/2)][\cos x + \sin(y/2)]}{\cos x - \sin(y/2)}$$
$$= \lim_{(x,y)\to(0,\pi)} [\cos x + \sin(y/2)] = \cos 0 + \sin(\pi/2) = 1 + 1 = 2 \neq 0 = g(0,\pi)$$

(b) **TRUE** The linearization is

$$\begin{aligned} L(x,y) &= f\left(\tfrac{\pi}{4}, \tfrac{\pi}{4}\right) + f_x\left(\tfrac{\pi}{4}, \tfrac{\pi}{4}\right)\left(x - \tfrac{\pi}{4}\right) + f_y\left(\tfrac{\pi}{4}, \tfrac{\pi}{4}\right)\left(y - \tfrac{\pi}{4}\right) \\ &= 2\cos\tfrac{\pi}{4}\sin\tfrac{\pi}{4} - 2\sin\tfrac{\pi}{4}\sin\tfrac{\pi}{4}\left(x - \tfrac{\pi}{4}\right) + 2\cos\tfrac{\pi}{4}\cos\tfrac{\pi}{4}\left(y - \tfrac{\pi}{4}\right) \\ &= 2\left(\tfrac{\sqrt{2}}{2}\right)\left(\tfrac{\sqrt{2}}{2}\right) - 2\left(\tfrac{\sqrt{2}}{2}\right)\left(\tfrac{\sqrt{2}}{2}\right)\left(x - \tfrac{\pi}{4}\right) + 2\left(\tfrac{\sqrt{2}}{2}\right)\left(\tfrac{\sqrt{2}}{2}\right)\left(y - \tfrac{\pi}{4}\right) \\ &= 1 - x + y \end{aligned}$$

(c) **FALSE** w never takes on negative values since it is the sum of squares.

(d) **TRUE**

$$\frac{\partial t}{\partial x} = \frac{\partial t}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial t}{\partial v}\frac{\partial v}{\partial x} = \left(\frac{1}{v}\right)(2x) - \left(\frac{u}{v^2}\right)(4y^3) = \frac{2x}{4xy^3} - \frac{x^2 - y^2}{16x^2y^6}(4y^3) = \frac{2x^2 - \left(x^2 - y^2\right)}{4x^2y^3} = \frac{x^2 + y^2}{4x^2y^3}$$

(e) **FALSE**

$$\nabla f(x,y) = \left\langle 3x^2, e^y \right\rangle \implies \nabla f(1,0) = \langle 3, 1 \rangle \implies \|\nabla f(1,0)\| = \sqrt{10}$$

The maximum rate of change of f at $(1,0)$ in any direction is $\sqrt{10}$ so there is no direction that will yield a rate of change greater than that.

■

2. [2350/102324 (16 pts)] You are provided the following information about the function $f(x,y)$. Use it to answer the questions below.

$$\begin{array}{llll} f(0,-1) = 0 & f_x(0,-1) = 0 & f_y(0,-1) = 0 & \\ f_{xx}(0,-1) = 1 & f_{xy}(0,-1) = -2 & f_{yy}(0,-1) = 0 & \\ f_{xxx}(x,y) = 1 & f_{xxy}(x,y) = 5x^2 & f_{xyy}(x,y) = -\dfrac{7}{y} & f_{yyy}(x,y) = 0 \end{array}$$

   (a) (6 pts) Find an appropriate quadratic Taylor polynomial for $f(x,y)$.

   (b) (4 pts) Estimate $f(0.1, -0.8)$.

   (c) (6 pts) Find an upper bound on the error in the approximation in part (b) if $|x| \leq 0.2$ and $|y+1| \leq 0.3$.

SOLUTION:

(a)

$$T_2(x,y) = f(0,-1) + f_x(0,-1)x + f_y(0,-1)(y+1) + \frac{1}{2!}\left[f_{xx}(0,-1)x^2 + 2f_{xy}(0,-1)x(y+1) + f_{yy}(0,-1)(y+1)^2\right]$$
$$= \frac{1}{2}\left[x^2 - 4x(y+1)\right]$$

(b)

$$f(0.1,-0.8) \approx T_2(0.1,-0.8) = \frac{1}{2}\left[(0.1)^2 - 4(0.1)(-0.8+1)\right] = -\frac{7}{200}$$

(c)

$$\begin{aligned}
|E(x,y)| &\le \frac{M}{3!}\left(|x| + |y+1|\right)^3 \\
M &= \max_{|x|\le 0.2, |y+1|\le 0.3}\left\{|f_{xxx}(x,y)|, |f_{xxy}(x,y)|, |f_{xyy}(x,y)|, |f_{yyy}(x,y)|\right\} \\
&= \max_{|x|\le 0.2, |y+1|\le 0.3}\left\{1, \left|5x^2\right|, \left|-\frac{7}{y}\right|, 0\right\} \\
&= \max\left\{1, \left|5(0.2)^2\right|, \left|\frac{-7}{-0.7}\right|, 0\right\} \\
&= \max\left\{1, \frac{1}{5}, 10, 0\right\} = 10 \\
|E(x,y)| &\le \frac{10}{6}(0.2+0.3)^3 = \frac{5}{3}\left(\frac{1}{2}\right)^3 = \frac{5}{24}
\end{aligned}$$

■

3. [2350/102324 (20 pts)] Consider the function $w(x,y,z) = 3z^2 + e^{3z+3} + \ln\left(\frac{1}{2}xy\right)$.

(a) [4 pts] What is the domain of w?

(b) [6 pts] Find the equation of the tangent plane to the level surface $w = 3$ at the point $(1,2,-1)$. Write your answer using the standard form $ax + by + cz = d$.

(c) [10 pts] Now suppose you are traveling along the path $\mathbf{r}(t) = e^t\,\mathbf{i} + (3-\cos t)\,\mathbf{j} + \left(t^2+t-1\right)\mathbf{k}$ where t represents time (minutes), $w(x,y,z)$ gives the amount (grams) of moisture in the air and distances are in km. Answer the following, assuming you are at the point $(x,y,z) = (1,2,-1)$. Include appropriate units.

i. With respect to time, is the moisture increasing, decreasing or not changing? If increasing or decreasing, at what rate?

ii. Find the rate of change of moisture with respect to distance.

SOLUTION:

(a) $\left\{(x,y,z) \in \mathbb{R}^3 \,\middle|\, xy > 0\right\}$

(b)

$$w_x = \frac{1}{xy/2}(y/2) = \frac{1}{x} \qquad w_y = \frac{1}{xy/2}(x/2) = \frac{1}{y} \qquad w_z = 6z + 3e^{3z+3}$$
$$\frac{1}{1}(x-1) + \frac{1}{2}(y-2) + \left[6(-1) + 3e^{3(-1)+3}\right](z+1) = 0 \implies x + \frac{1}{2}y - 3z = 5$$

(c) You are at the given point in space when $t = 0$ and $\mathbf{r}'(t) = e^t\,\mathbf{i} + \sin t\,\mathbf{j} + (2t+1)\,\mathbf{k}$.

i.

$$\left.\frac{dw}{dt}\right|_{t=0} = \nabla w(1,2,-1)\cdot\mathbf{r}'(0) = \left\langle 1, \frac{1}{2}, -3\right\rangle \cdot \langle 1,0,1\rangle = -2$$

The moisture is decreasing at a rate of two grams per minute.

ii.

$$\left.\frac{\mathrm{d}w}{\mathrm{d}s}\right|_{(1,2,-1)} = \nabla w(1,2,-1)\cdot\frac{\mathbf{r}'(0)}{\|\mathbf{r}'(0)\|} = \left\langle 1,\frac{1}{2},-3\right\rangle\cdot\frac{\langle 1,0,1\rangle}{\sqrt{1+0+1}} = -\sqrt{2}\,\frac{\text{gram}}{\text{km}}$$

■

4. [2350/102324 (34 pts)] You need to find the dimensions that minimize the total cost of material needed to construct a rectangular box having the following characteristics: (i) it is open-topped (has five sides); (ii) its volume is 600 cubic inches; (iii) the material for its bottom costs 6¢ per square inch; (iv) the material for the sides costs 5¢ per square inch; (v) l, w, h are the box's length, width and height, respectively. Perform the following steps to do this problem. Do not use Lagrange Multipliers and include units in your answer.

   (a) [8 pts] Show that the cost function can be written as $C(l,w) = 6000\left(l^{-1}+w^{-1}\right)+6lw$.

   (b) [12 pts] Find l, w that minimize the cost function.

   (c) [8 pts] Verify that the l, w you found in part (b) do result in a minimum of the cost function.

   (d) [3 pts] What are the dimensions of the cheapest box?

   (e) [3 pts] How much will it cost to build the box?

SOLUTION:

(a) The cost is given by cost (¢) $= 2\left(5\frac{¢}{\text{in}^2}\right)(lh)\text{in}^2 + 2\left(5\frac{¢}{\text{in}^2}\right)(wh)\text{in}^2 + \left(6\frac{¢}{\text{in}^2}\right)(lw)\text{in}^2$. Since $lwh = 600 \implies h = \dfrac{600}{lw}$ we have

$$C(l,w) = 10l\left(\frac{600}{lw}\right) + 10w\left(\frac{600}{lw}\right) + 6lw = \frac{6000}{w} + \frac{6000}{l} + 6lw = 6000\left(l^{-1}+w^{-1}\right)+6lw$$

(b)

$$C_l = -6000l^{-2} + 6w = 0 \implies w = \frac{1000}{l^2}$$

$$C_w = -6000w^{-2} + 6l = 0 \implies l = \frac{1000}{w^2}$$

$$w = \frac{1000}{(1000/w^2)^2} \implies \frac{w^4}{1000} - w = 0 \implies w\left(\frac{w^3}{1000}-1\right) = 0 \implies w = 0, 10$$

Since the domain of $C(l,w)$ does not contain $w=0$, we choose $w = 10 \implies l = \dfrac{1000}{10^2} = 10$

(c) Use the Second Derivatives Test.

$$C_{ll} = 12000l^{-3} \implies C_{ll}(10,10) = 12000/1000 = 12$$

$$C_{ww} = 12000w^{-3} \implies C_{ww}(10,10) = 12000/1000 = 12$$

$$C_{lw} = 6$$

$$C_{ll}(10,10)C_{ww}(10,10) - C_{lw}^2(10,10) = 108 \text{ and } C_{ll}(10,10) > 0 \implies C(10,10) \text{ is a local minimum}$$

(d) The dimensions of the box are $l\times w\times h = 10$ in $\times$ 10 in $\times$ 6 in

(e) The cost to manufacture the box is $C(10,10) = 6000\,(0.1+0.1) + 6(10)(10) = 1800$¢ $= \$18$

■

5. [2350/102324 (20 pts)] The satisfaction you get from eating x slices of bacon, y eggs and z potatoes is given by the satisfaction function $s(x,y,z) = \sqrt{xyz}$. Suppose you have six dollars to spend and a rather hearty appetite. If each slice of bacon costs \$0.50, each egg costs \$0.25, and each potato costs \$0.10, use Lagrange Multipliers to determine how many bacon slices, eggs and potatoes you should purchase to be the most satisfied.

SOLUTION:

We need to optimize $s(x,y,z)$ subject to the constraint $g(x,y,z) = \frac{1}{2}x + \frac{1}{4}y + \frac{1}{10}z = 6$ with x, y, z all nonnegative. Note that the constraint can also be written as $50x + 25y + 10z = 600$. Lagrange multipliers technique gives

$$s_x = \frac{yz}{2\sqrt{xyz}} = \frac{\sqrt{yz}}{2\sqrt{x}} \qquad\qquad g_x = \frac{1}{2}$$

$$s_y = \frac{xz}{2\sqrt{xyz}} = \frac{\sqrt{xz}}{2\sqrt{y}} \qquad\qquad g_y = \frac{1}{4}$$

$$s_z = \frac{xy}{2\sqrt{xyz}} = \frac{\sqrt{xy}}{2\sqrt{z}} \qquad\qquad g_z = \frac{1}{10}$$

yielding the following system of nonlinear equations

$$\frac{\sqrt{yz}}{2\sqrt{x}} = \frac{1}{2}\lambda \implies \frac{\sqrt{yz}}{\sqrt{x}} = \lambda \tag{1}$$

$$\frac{\sqrt{xz}}{2\sqrt{y}} = \frac{1}{4}\lambda \implies \frac{2\sqrt{xz}}{\sqrt{y}} = \lambda \tag{2}$$

$$\frac{\sqrt{xy}}{2\sqrt{z}} = \frac{1}{10}\lambda \implies \frac{5\sqrt{xy}}{\sqrt{z}} = \lambda \tag{3}$$

$$\frac{1}{2}x + \frac{1}{4}y + \frac{1}{10}z = 6 \tag{4}$$

Note that none of x, y or z can be zero in this system, since $s(x,y,z)$ is not differentiable for those values. Combining (1) and (2) yields

$$\frac{\sqrt{yz}}{\sqrt{x}} = \frac{2\sqrt{xz}}{\sqrt{y}} \implies y = 2x$$

while combining (1) and (3) yields

$$\frac{\sqrt{yz}}{\sqrt{x}} = \frac{5\sqrt{xy}}{\sqrt{z}} \implies z = 5x$$

Using this information in (4) gives

$$\frac{1}{2}x + \frac{1}{4}(2x) + \frac{1}{10}(5x) = 6 \implies \frac{3}{2}x = 6 \implies x = 4$$

from which we conclude that $y = 8$ and $z = 20$. To maximize our satisfaction we should buy 4 slices of bacon, 8 eggs and 20 potatoes.

■