1. [2350/092524 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

   (a) For any path $\mathbf{r}(t)$, the acceleration, $\mathbf{a}(t)$, and the binormal, $\mathbf{B}(t)$, are always orthogonal.

   (b) For nonzero vectors $\mathbf{A}$ and $\mathbf{B}$, $(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{A}+\mathbf{B})=0$.

   (c) $\text{comp}_{\mathbf{j}}\,(2\,\mathbf{j}+\mathbf{k})=1$.

   (d) $2x^2+12x=y^2-4z-18$ is an elliptic paraboloid.

   (e) The equation of the osculating plane of the path $\mathbf{r}(t)=e^t\,\mathbf{j}+\ln t\,\mathbf{k}, t>0$, is $x=0$.

   SOLUTION:

   (a) **TRUE** The acceleration always lies in the osculating plane whose normal is $\mathbf{B}(t)$.

   (b) **TRUE** $\mathbf{A}\times\mathbf{B}$ is orthogonal to both $\mathbf{A}$ and $\mathbf{B}$.

   (c) **FALSE** $\text{comp}_{\mathbf{j}}\,(2\,\mathbf{j}+\mathbf{k})=\dfrac{(2\,\mathbf{j}+\mathbf{k})\cdot\mathbf{j}}{\|\mathbf{j}\|}=\dfrac{2+0}{1}=2$

   (d) **FALSE** It is a hyperbolic paraboloid. $2x^2+12x=y^2-4z-18 \iff z=-\left(\dfrac{x+3}{\sqrt{2}}\right)^2+\left(\dfrac{y}{2}\right)^2$

   (e) **TRUE** The path lies in the yz-plane whose equation is $x=0$

2. [2350/092524 (24 pts)] Consider the points $P(-1,5,3)$ and $Q(6,2,-2)$.

   (a) (8 pts) Find an equation for the set of all points that are equidistant from P and Q. Simplify your answer completely and name the surface.

   (b) (8 pts) Find the work done by $\mathbf{F}=5\,\mathbf{i}-3\,\mathbf{j}+7\,\mathbf{k}$ moving an object from P to Q.

   (c) (8 pts) Suppose an unknown force, $\mathbf{F}$, applied at the point Q and normal to $\overrightarrow{PQ}$, produces a torque of magnitude $9\sqrt{83}$ about point P. What is the magnitude of the force?

   SOLUTION:

   (a) Let (x,y,z) be an arbitrary point on the surface. Then

$$\sqrt{(x+1)^2+(y-5)^2+(z-3)^2}=\sqrt{(x-6)^2+(y-2)^2+(z+2)^2}$$
$$x^2+2x+1+y^2-10y+25+z^2-6z+9=x^2-12x+36+y^2-4y+4+z^2+4z+4$$
$$14x-6y-10z=9$$

   This is a plane.

   (b)

$$\text{Displacement}=\mathbf{D}=\overrightarrow{PQ}=\langle 6-(-1),2-5,-2-3\rangle=7\,\mathbf{i}-3\,\mathbf{j}-5\,\mathbf{k}$$
$$\text{Work}=\mathbf{F}\cdot\mathbf{D}=(5\,\mathbf{i}-3\,\mathbf{j}+7\,\mathbf{k})\cdot(7\,\mathbf{i}-3\,\mathbf{j}-5\,\mathbf{k})=9$$

   (c)

$$\|\boldsymbol{\tau}\|=\|\mathbf{F}\|\|\overrightarrow{PQ}\|\sin\theta \implies \|\mathbf{F}\|=\frac{\|\boldsymbol{\tau}\|}{\|\overrightarrow{PQ}\|\sin\theta}=\frac{9\sqrt{83}}{\sqrt{7^2+(-3)^2+(-5)^2}\sin 90^\circ}=\frac{9\sqrt{83}}{\sqrt{83}(1)}=9$$

3. [2350/092524 (22 pts)] You are playing indoors with a paper airplane, the velocity of which is given by $\mathbf{v}(t)=(1-3\cos t)\,\mathbf{i}+3\sin t\,\mathbf{k}$.

   (a) [6 pts] Find the tangential component of the acceleration vector.

   (b) [10 pts] If the initial position of the airplane is $\mathbf{r}(0)=\mathbf{i}-\mathbf{j}+\mathbf{k}$, find an expression for the path of the airplane.

(c) [6 pts] Look out! On its maiden voyage the airplane crashes into the ceiling, considered as a plane parallel to the xy-plane. The crash occurs when $t = \pi/2$. Find the acute angle at which the airplane hits the ceiling.

SOLUTION:

(a)

$$a_T = \frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}\| = \frac{\mathrm{d}}{\mathrm{d}t}\sqrt{1 - 6\cos t + 9\cos^2 t + 9\sin^2 t} = \frac{\mathrm{d}}{\mathrm{d}t}\sqrt{10 - 6\cos t} = \frac{3\sin t}{\sqrt{10 - 6\cos t}}$$

Alternatively

$$a_T = \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{\|\mathbf{r}'(t)\|} = \frac{\mathbf{v}(t)\cdot\mathbf{v}'(t)}{\|\mathbf{v}(t)\|} = \frac{[(1-3\cos t)\,\mathbf{i} + 3\sin t\,\mathbf{k}]\cdot(3\sin t\,\mathbf{i} + 3\cos t\,\mathbf{k})}{\sqrt{(1-3\cos t)^2 + 9\sin^2 t}} = \frac{3\sin t}{\sqrt{10 - 6\cos t}}$$

(b)

$$\mathbf{r}(t) = \int \mathbf{v}(t)\,\mathrm{d}t = \left[\int (1 - 3\cos t)\,\mathrm{d}t\right]\mathbf{i} + \left[\int 3\sin t\,\mathrm{d}t\right]\mathbf{k} = (t - 3\sin t + c_1)\,\mathbf{i} + c_2\,\mathbf{j} + (-3\cos t + c_3)\,\mathbf{k}$$

$$\mathbf{r}(0) = c_1\,\mathbf{i} + c_2\,\mathbf{j} + (-3 + c_3)\,\mathbf{k} = \mathbf{i} - \mathbf{j} + \mathbf{k} \implies c_1 = 1,\ c_2 = -1,\ c_3 = 4$$

$$\implies \mathbf{r}(t) = (t - 3\sin t + 1)\,\mathbf{i} - \mathbf{j} + (4 - 3\cos t)\,\mathbf{k}$$

(c) A vector normal to the ceiling is $\mathbf{k}$ and a vector in the direction of motion at the point of impact is

$$\mathbf{r}'(\pi/2) = \mathbf{v}(\pi/2) = \mathbf{i} + 3\,\mathbf{k}$$

giving the angle between these as

$$\theta = \cos^{-1}\frac{\mathbf{v}(\pi/2)\cdot\mathbf{k}}{\|\mathbf{v}(\pi/2)\|\|\mathbf{k}\|} = \cos^{-1}\frac{3}{\sqrt{10}}$$

The the angle at which the airplane hits the ceiling is $\dfrac{\pi}{2} - \cos^{-1}\dfrac{3}{\sqrt{10}}$. Alternatively, using $-\mathbf{k}$ as the normal to the ceiling yields the angle of impact as $\cos^{-1}\dfrac{-3}{\sqrt{10}} - \dfrac{\pi}{2}$.

4. [2350/092524 (20 pts)] A mosquito is buzzing along the path $\mathbf{r}(t) = t\,\mathbf{i} + \frac{2\sqrt{2}}{3}t^{3/2}\,\mathbf{j} + \frac{1}{2}t^2\,\mathbf{k},\ t \geq 0$, with distance measured in yards and time in seconds.

(a) [10 pts] Starting from time $t = 0$, how long does it take the mosquito to travel 12 yards along this path?

(b) [5 pts] After 1 second, how far is the mosquito from its starting position at $t = 0$?

(c) [5 pts] Now suppose that after flying along the aforementioned path for 2 seconds, the mosquito notices a bat approaching from behind. In an effort to avoid becoming the bat's dinner, from this point the mosquito leaves the original path, flying straight ahead along a line at a constant speed of 1 yard/sec. Find the mosquito's coordinates after it travels along this line for 3 seconds.

SOLUTION:

(a)

$$\mathbf{r}'(t) = \mathbf{i} + \sqrt{2t}\,\mathbf{j} + t\,\mathbf{k} \implies \|\mathbf{r}'(t)\| = \sqrt{1 + 2t + t^2} = \sqrt{(1+t)^2} = |1 + t| = 1 + t \text{ since } t \geq 0$$

$$L = 12 = \int_0^c \|\mathbf{r}'(t)\|\,\mathrm{d}t = \int_0^c (1+t)\,\mathrm{d}t = \left(t + \frac{t^2}{2}\right)\Big|_0^c = c + \frac{c^2}{2}$$

$$\implies c^2 + 2c - 24 = (t+6)(t-4) = 0 \implies t = -6, 4$$

Since we are only interested in positive t values, it takes the mosquito 4 seconds to fly 12 yards.

(b) After 1 second, the mosquito is at the point $\left(1, \frac{2\sqrt{2}}{3}, \frac{1}{2}\right)$. It was initially at the origin so the distance from the starting point is

$$d = \sqrt{1^2 + \left(\frac{2\sqrt{2}}{3}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{1 + \frac{8}{9} + \frac{1}{4}} = \sqrt{\frac{36 + 32 + 9}{36}} = \frac{\sqrt{77}}{6} \text{ yards}$$

(c) After 2 seconds, the mosquito's position vector is $\mathbf{r}(2) = 2\,\mathbf{i} + \frac{8}{3}\,\mathbf{j} + 2\,\mathbf{k}$ and the tangent vector there is $\mathbf{r}'(2) = \mathbf{i} + 2\,\mathbf{j} + 2\,\mathbf{k}$. Since the speed must be one, we find a unit vector in the direction of the tangent vector and use the arclength parameterization of the line to obtain

$$\mathbf{r}_{\text{line}}(s) = \left\langle 2, \tfrac{8}{3}, 2 \right\rangle + s\frac{\langle 1, 2, 2\rangle}{\sqrt{1^2 + 2^2 + 2^2}} = \left\langle 2, \tfrac{8}{3}, 2 \right\rangle + \frac{s}{3}\langle 1, 2, 2\rangle$$

After 3 seconds flying on the line, the mosquito's position vector is (coordinates are)

$$\mathbf{r}_{\text{line}}(3) = \left\langle 2, \tfrac{8}{3}, 2 \right\rangle + \tfrac{3}{3}\langle 1, 2, 2\rangle = \left\langle 3, \tfrac{14}{3}, 4 \right\rangle \quad \text{or} \quad \left(3, \tfrac{14}{3}, 4\right)$$

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5. [2350/092524 (24 pts)] A bee leaves its hive located in a tree at the point $A(1, 2, 3)$ and flies in a straight line to a sunflower located at point $B(-1, 4, 7)$. From there it flies in a straight line to a daisy at point $C(2, -5, -1)$ and returns on a straight path back to the hive.

(a) (8 pts) Find the area of the triangular region inside the bee's path.

(b) (8 pts) Find the equation of the plane containing the bee's path. Write your answer in the form $ax + by + cz = d$.

(c) (8 pts) The queen bee leaves the hive and crawls along a straight branch to point D at the origin. Find the volume of the parallelepiped formed by the queen bee's path, and the bee's path from the hive to the sunflower and from the hive to the daisy.

SOLUTION:

(a) The vector from the hive to the sunflower is $\overrightarrow{AB} = \langle -2, 2, 4\rangle$ and the vector from the hive to the daisy is $\overrightarrow{AC} = \langle 1, -7, -4\rangle$. These two vectors form a parallelogram. The area of the triangular region inside the bee's path is one-half the area of this parallelogram, that is,

$$\text{Area} = \frac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\| = \frac{1}{2}\left\| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 2 & 4 \\ 1 & -7 & -4 \end{vmatrix} \right\| = \frac{1}{2}\|20\,\mathbf{i} - 4\,\mathbf{j} + 12\,\mathbf{k}\| = \frac{\sqrt{560}}{2} = 2\sqrt{35}$$

Note that you can use any two of the vectors that make up the bee's path to find the requested area.

(b) To find the equation of a plane we need a normal vector and a point in the plane. We already have a normal from part (a) as $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = 20\,\mathbf{i} - 4\,\mathbf{j} + 12\,\mathbf{k}$. Using point A as a point in the plane gives

$$20(x-1) - 4(y-2) + 12(z-3) = 0 \implies 20x - 4y + 12z = 48 \;\text{ or }\; 5x - y + 3z = 12$$

Again, note that any of the vectors making up the bee's path and any of the points the bee visited could be used in the above calculation.

(c) The vector giving the queen bee's path is $\overrightarrow{AD} = \langle -1, -2, -3\rangle$ and we have the other two vectors from earlier. The volume is given by the absolute value of the scalar triple product of the three vectors (taken in any order) as

$$\text{Volume} = |\overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{AC})| = \left| \begin{vmatrix} -1 & -2 & -3 \\ -2 & 2 & 4 \\ 1 & -7 & -4 \end{vmatrix} \right| = |-48| = 48$$

Alternatively, using previous computations,

$$\text{Volume} = |\overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{AC})| = |\langle -1, -2, -3\rangle \cdot \langle 20, -4, 12\rangle| = |-48| = 48$$

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