1. [2350/121923 (24 pts)] Write the word **TRUE** or **FALSE** as appropriate. Write your answers in a single column separate from any work you do. No work need be shown. No partial credit given.

   (a) If $\mathbf{a} = \mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k},\ \mathbf{b} = -\mathbf{i} + 2\,\mathbf{k}$ and $\mathbf{c} = -3\,\mathbf{j} + \mathbf{k}$, then $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 1$.

   (b) The first order Taylor polynomial for $h(x, y) = e^{-x^2-y^2}$ centered at $(0, 0)$ is $T_1(x, y) = 1$.

   (c) The curvature of the path $\mathbf{r}(s)$ given by the arc length parameterization $\mathbf{r}(s) = \dfrac{\sqrt{3}}{2}s\,\mathbf{i} + \sin\dfrac{s}{2}\,\mathbf{j} - \cos\dfrac{s}{2}\,\mathbf{k}$ is always $\kappa = 0.25$.

   (d) If $f(x, y, z) = \ln\left(x^2 + y^3 + z^4\right)$, there exists a unit vector $\mathbf{w}$ such that the instantaneous rate of change of f with respect to distance in the direction of $\mathbf{w}$ at the point $(1, -1, 1)$ equals 10.

   (e) The equations of the normal line and tangent plane to the surface $x^2 + y^2 - z^2 = 4$ at the point $P(2, 1, -1)$ are, respectively, $\mathbf{l}(t) = \langle 4t + 2, 2t + 1, 2t - 1\rangle$, t a real number, and $2x + y + z = 4$.

   (f) Particles moving along straight line paths always experience zero acceleration.

SOLUTION:

(a) **FALSE**

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 0 & -3 & 1 \end{vmatrix} = 17$$

(b) **TRUE**

$$T_1(x, y) = h(0, 0) + h_x(0, 0)(x - 0) + h_y(0, 0)(y - 0) = 1 - \left(2xe^{-x^2-y^2}\right)\Big|_{(0,0)} x - \left(2ye^{-x^2-y^2}\right)\Big|_{(0,0)} y = 1$$

(c) **TRUE** Since this is an arc length parameterization, $\|\mathbf{r}'(s)\| = 1$ so that

$$\mathbf{T}(s) = \mathbf{r}'(s) = \frac{\sqrt{3}}{2}\,\mathbf{i} + \frac{1}{2}\cos\frac{s}{2}\,\mathbf{j} + \frac{1}{2}\sin\frac{s}{2}\,\mathbf{k}$$

$$\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} = -\frac{1}{4}\sin\frac{s}{2}\,\mathbf{j} + \frac{1}{4}\cos\frac{s}{2}\,\mathbf{k}$$

$$\kappa = \left\|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\right\| = \sqrt{\left(-\frac{1}{4}\sin\frac{s}{2}\right)^2 + \left(\frac{1}{4}\cos\frac{s}{2}\right)^2} = \frac{1}{4}$$

Alternatively, $\kappa = \dfrac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3} = \|\mathbf{r}' \times \mathbf{r}''\|$ since $\|\mathbf{r}'\| = 1$ due to the fact that this is an arc length parameterization. Thus

$$\kappa = \left\| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sqrt{3}/2 & \frac{1}{2}\cos\frac{s}{2} & \frac{1}{2}\sin\frac{s}{2} \\ 0 & -\frac{1}{4}\sin\frac{s}{2} & \frac{1}{4}\cos\frac{s}{2} \end{vmatrix} \right\| = \left\| \left(\frac{1}{8}\cos^2\frac{s}{2} + \frac{1}{8}\sin^2\frac{s}{2}\right)\mathbf{i} - \frac{\sqrt{3}}{8}\cos\frac{s}{2}\,\mathbf{j} - \frac{\sqrt{3}}{8}\sin\frac{s}{2}\,\mathbf{k} \right\| = \sqrt{\frac{1}{64} + \frac{3}{64}} = \frac{1}{4}$$

(d) **FALSE** The maximum rate of change of f at the point in question is $\|\nabla f(1, -1, 1)\|$.

$$\nabla f(x, y, z) = \left\langle \frac{2x}{x^2 + y^3 + z^4}, \frac{3y^2}{x^2 + y^3 + z^4}, \frac{4z^3}{x^2 + y^3 + z^4} \right\rangle$$

$$\nabla f(1, -1, 1) = \langle 2, 3, 4\rangle$$

$$\|\nabla f(1, -1, 1)\| = \sqrt{2^2 + 3^2 + 4^2} = \sqrt{29}$$

Since $10 > \sqrt{29}$, there is no direction that will provide a rate of change of f with respect to distance at the point $(1, -1, 1)$ equal to 10.

(e) **TRUE** With $G(x, y, z) = x^2 + y^2 - z^2, \nabla G = \langle 2x, 2y, -2z \rangle \implies \nabla G(P) = \langle 4, 2, 2 \rangle$. Then

normal line: $\langle 2, 1, -1 \rangle + t\langle 4, 2, 2 \rangle = \langle 4t + 2, 2t + 1, 2t - 1 \rangle$

tangent plane: $4(x - 2) + 2(y - 1) + 2(z + 1) = 0 \implies 2x - 4 + y - 1 + z + 1 = 0 \implies 2x + y + z = 4$

(f) **FALSE** They never experience a normal component of acceleration since their direction never changes. However, they can experience a nonzero tangential acceleration.

■

2. [2350/121923 (20 pts)] You are going on a trip and want to maximize the space in your carry-on suitcase, assumed to be in the shape of a rectangular box. Find the dimensions of the suitcase of maximum volume if the sum of the suitcase's width (w), height (h) and two times its length (l) is 120 cm. No credit for using Lagrange Multipliers. Instead, solve this as a two dimensional optimization problem, verifying your result using the Second Derivatives Test.

SOLUTION:

With the dimensions of the suitcase being w, h, and l, $w + h + 2l = 120$. The suitcase's volume is $V(l, w, h) = lwh$. Combining these we get $V(l, w) = lw(120 - 2l - w) = 120lw - 2l^2w - lw^2$. (Note: The volume could have been written as a function of h and w or h and l.)

$$V_l = 120w - 4lw - w^2 = w(120 - 4l - w) = 0 \tag{1}$$

$$V_w = 120l - 2l^2 - 2lw = l(120 - 2l - 2w) = 0 \tag{2}$$

We can safely assume that $w > 0$ and $l > 0$, otherwise no suitcase would exist. Solving the parenthetical portion of Eq. (1) for w and using this in the parenthetical portion of Eq. (2) yields

$$w = 120 - 4l$$

$$120 - 2l - 2(120 - 4l) = 0$$

$$-120 + 6l = 0$$

$$l = 20 \implies w = 120 - 80 = 40 \implies h = 120 - 2(20) - 40 = 40$$

Apply the Second Derivatives Test to classify the critical point $(l, w) = (20, 40)$.

$$V_{ll} = -4w$$

$$V_{ww} = -2l$$

$$V_{lw} = V_{wl} = 120 - 4l - 2w$$

$$D(20, 40) = V_{ll}(20, 40)V_{ww}(20, 40) - [V_{lw}(20, 40)]^2$$

$$= [-4(40)(-2)(20)] - [120 - 4(20) - 2(40)]^2 = (160)(40) - (-40)^2$$

$$= 40(160 - 40) > 0 \text{ and } V_{ll}(20, 40) = -4(40) < 0$$

so that the critical point gives a local maximum. The suitcase's dimensions are $l \times w \times h = 20 \times 40 \times 40$ cm.

■

3. [2350/121923 (25 pts)] The Hundred Acre Wood is under the influence of the vector field $\mathbf{F} = e^y\,\mathbf{i} + (xe^y + \sin z)\,\mathbf{j} + y\cos z\,\mathbf{k}$. Consider the points $M(0, 1, 0)$ and $N(\pi, 3, 2\pi)$ in the forest.

(a) (8 pts) Winnie the Pooh waddles along the path $\mathbf{r}(t) = \left[\frac{\pi}{2}(t - 1)\right]\mathbf{i} + t\,\mathbf{j} + \pi\,(t - 1)\,\mathbf{k}$ between the points. Set up and simplify, but **do not evaluate**, the integral that will directly compute the amount of work Pooh does in waddling along the path from point M to point N. Pooh's friends Kanga and Roo offer to help you with the simplification by telling you that $\sin[\pi(t - 1)] = -\sin\pi t$ and $\cos[\pi(t - 1)] = -\cos\pi t$.

(b) (14 pts) Pooh's gloomy friend Eeyore watches as Tigger bounces along the path $(t - \sin 4t)\,\mathbf{i} + (2 - \cos 3t)\,\mathbf{j} + 2t\,\mathbf{k},\ 0 \le t \le \pi$ and Owl flies along the path $\sqrt{\frac{t^3}{8\pi}}\,\mathbf{i} + \left(1 + \frac{t}{\pi}e^{\pi - t/2}\right)\mathbf{j} + t\,\mathbf{k},\ 0 \le t \le 2\pi$. Both of these paths begin at point M and end at point N. Eeyore claims that both Tigger and Owl did the same amount of work, W, as Pooh did and announces that an important Calculus 3 theorem can be used to compute W. Show that Eeyore is correct and use the theorem to find W. Hint: No difficult integration is required.

(c) (3 pts) Finally, Pooh's other acquaintance, Gopher, crawls from point N to the point $(2, 3, 50)$ then tunnels underground back to point N. How much work did he do?

SOLUTION:

(a) Note that t runs from 1 to 3.

$$\mathbf{r}'(t) = \left\langle \frac{\pi}{2}, 1, \pi \right\rangle$$

$$\mathbf{F}(\mathbf{r}(t)) = \left\langle e^t, \frac{\pi}{2}(t-1)e^t + \sin\pi(t-1), t\cos\pi(t-1) \right\rangle$$

$$\begin{aligned}\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) &= \left\langle e^t, \frac{\pi}{2}(t-1)e^t + \sin\pi(t-1), t\cos\pi(t-1) \right\rangle \cdot \left\langle \frac{\pi}{2}, 1, \pi \right\rangle \\ &= \frac{\pi}{2}e^t + \frac{\pi}{2}(t-1)e^t + \sin\pi(t-1) + \pi t\cos\pi(t-1) \\ &= \frac{\pi}{2}te^t - \sin\pi t - \pi t\cos\pi t\end{aligned}$$

$$\text{Work} = \int_1^3 \left(\frac{\pi}{2}te^t - \sin\pi t - \pi t\cos\pi t\right) \mathrm{d}t$$

(b) The fact the work is the same along the three paths suggests that the vector field may be conservative. It's defined throughout $\mathbb{R}^3$, a simply connected region. Moreover,

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^y & xe^y + \sin z & y\cos z \end{vmatrix} = \cos z\,\mathbf{i} + 0\,\mathbf{j} + e^y\,\mathbf{k} - (e^y\,\mathbf{k} + \cos z\,\mathbf{i} + 0\,\mathbf{j}) = \mathbf{0}$$

Thus, $\mathbf{F}$ is conservative and consequently can be written as $\mathbf{F} = \nabla f$ for some potential function f. We find f.

$$f(x, y, z) = \int e^y \, \mathrm{d}x = xe^y + g(y, z)$$

$$f_y = xe^y + g_y(y, z) = xe^y + \sin z \implies g_y(y, z) = \sin z \implies g(y, z) = \int \sin z \, \mathrm{d}y = y\sin z + h(z)$$

$$f(x, y, z) = xe^y + y\sin z + h(z)$$

$$f_z = y\cos z + h'(z) = y\cos z \implies h'(z) = 0 \implies h(z) = c$$

$$f(x, y, z) = xe^y + y\sin z + c$$

We can now use the Fundamental Theorem of Line Integrals to find the work as

$$\text{Work} = \int_{(0,1,0)}^{(\pi,3,2\pi)} \nabla f \cdot \mathrm{d}\mathbf{r} = f(\pi, 3, 2\pi) - f(0, 1, 0) = \pi e^3 + 3\sin 2\pi + c - \left(0e^1 + 1\sin 0 + c\right) = \pi e^3$$

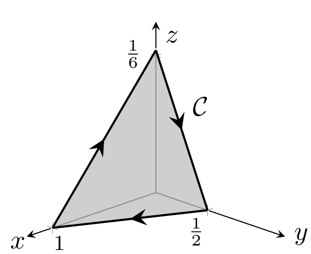
(c) Since the field is conservative and Gopher's path is closed, he does 0 net work.

■

4. [2350/121923 (20 pts)] Use Stokes' Theorem to find the circulation of $\mathbf{V} = -2yz\,\mathbf{j} + 6yz\,\mathbf{k}$ around $\mathcal{C}$, the intersection of the plane $x + 2y + 6z = 1$ with the first octant, oriented clockwise when viewed from above.

SOLUTION:

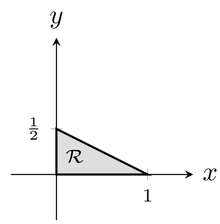
Here is a sketch of the curve and the surface.



$$\nabla\times\mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & -2yz & 6yz \end{vmatrix} = (6z+2y)\,\mathbf{i}$$

$$g(x,y,z) = x+2y+6z \implies \nabla g = \langle 1,2,6\rangle \quad \text{use } -\nabla g \text{ for proper orientation}$$

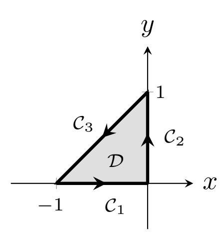
$$\text{project onto the } xy\text{-plane} \implies \mathbf{p}=\mathbf{k},\ |\nabla g\cdot\mathbf{p}| = 6 \text{ with } \mathcal{R} \text{ shown below}$$



$$\begin{aligned}
\text{Circulation} &= \oint_{\mathcal{C}} \mathbf{V}\cdot\mathrm{d}\mathbf{r} = \iint_{\mathcal{S}} (\nabla\times\mathbf{V})\cdot\mathbf{n}\,\mathrm{d}S = \iint_{\mathcal{R}} \nabla\times\mathbf{V}\cdot\left(\frac{-\nabla g}{|\nabla g\cdot\mathbf{k}|}\right)\mathrm{d}A \\
&= \iint_{\mathcal{R}} \langle 2y+6z,0,0\rangle\cdot\frac{\langle -1,-2,-6\rangle}{6}\,\mathrm{d}A \\
&= -\frac{1}{6}\iint_{\mathcal{R}} (2y+6z)\ \mathrm{d}A \qquad \text{(use surface to eliminate } z\text{)} \\
&= -\frac{1}{6}\int_0^1\int_0^{(1-x)/2} (1-x)\,\mathrm{d}y\,\mathrm{d}x \\
&= -\frac{1}{12}\int_0^1 (1-x)^2\ \mathrm{d}x \\
&= \frac{1}{36}(1-x)^3\Big|_0^1 = -\frac{1}{36}
\end{aligned}$$

Alternatively, Circulation $= -\dfrac{1}{6}\displaystyle\int_0^{1/2}\int_0^{1-2y} (1-x)\,\mathrm{d}x\,\mathrm{d}y$. Furthermore, one could also project onto either the xz- or yz-plane. ■

5. [2350/121923 (33 pts)] Consider the vector field $\mathbf{F}(x,y) = x^2y\,\mathbf{i} + xy^2\,\mathbf{j}$ and the following figure. As shown below, $\mathcal{D}$ has oriented boundary $\partial\mathcal{D} = \mathcal{C}_1\cup\mathcal{C}_2\cup\mathcal{C}_3$.



(a) (6 pts) Compute $\mathbf{k}\cdot\nabla\times\mathbf{F}$ and determine if this function attains an absolute maximum and minimum value on the region $\mathcal{D}\cup\partial\mathcal{D}$. Do not find these values, if they exist, simply justify your answer in a few words.

(b) (3 pts) Without doing any integration, explain in words why there is no flux through or flow along $\mathcal{C}_1$ and $\mathcal{C}_2$.

(c) (10 pts) Using the parameterization $x=-t, y=-t+1$, evaluate $\displaystyle\int_{\mathcal{C}_3} P\,\mathrm{d}y - Q\,\mathrm{d}x$.

(d) (4 pts) Using your answers to parts (b) and (c), what is $\displaystyle\int_{\partial\mathcal{D}} \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}s$? Hint: no integration is required.

(e) (10 pts) Find the flux through $\partial\mathcal{D}$ using Green's Theorem.

SOLUTION:

(a)

$$\mathbf{k}\cdot\nabla\times\mathbf{F}=\mathbf{k}\cdot\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ x^2y & xy^2 & 0\end{vmatrix}=\mathbf{k}\cdot\left(y^2-x^2\right)\mathbf{k}=y^2-x^2=\frac{\partial Q}{\partial y}-\frac{\partial P}{\partial y}$$

Since y^2-x^2 is continuous everywhere, it is continuous on $\mathcal{D}\cup\partial D$, a closed, bounded set and thus an absolute maximum and minimum value of $\mathbf{k}\cdot\nabla\times\mathbf{F}$ are guaranteed to exist by the Extreme Value Theorem.

(b) $\mathbf{F}=\mathbf{0}$ on both curves.

(c) To trace out $\mathcal{C}_3$, we need $0\le t\le 1$. $P=x^2y=(-t)^2(-t+1)=-t^3+t^2$, $Q=xy^2=-t(-t+1)^2=-t^3+2t^2-t$, $\mathrm{d}x=-\mathrm{d}t$ and $\mathrm{d}y=-\mathrm{d}t$. Then

$$\begin{aligned}\int_{\mathcal{C}_3}P\,\mathrm{d}y-Q\,\mathrm{d}x&=\int_0^1\left(-t^3+t^2\right)(-\mathrm{d}t)-\left(-t^3+2t^2-t\right)(-\mathrm{d}t)\\&=\int_0^1\left(t^2-t\right)\mathrm{d}t=\left(\frac{1}{3}t^3-\frac{1}{2}t^2\right)\Big|_0^1=-\frac{1}{6}\end{aligned}$$

(d)

$$\int_{\partial\mathcal{D}}\mathbf{F}\cdot\mathbf{n}\,\mathrm{d}s=\int_{\mathcal{C}_1}\mathbf{F}\cdot\mathbf{n}\,\mathrm{d}s+\int_{\mathcal{C}_2}\mathbf{F}\cdot\mathbf{n}\,\mathrm{d}s+\int_{\mathcal{C}_3}\mathbf{F}\cdot\mathbf{n}\,\mathrm{d}s=0+0-\frac{1}{6}=-\frac{1}{6}$$

(e)

$$\begin{aligned}\text{Flux}&=\iint_{\mathcal{D}}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)\mathrm{d}A=\int_{-1}^0\int_0^{x+1}4xy\,\mathrm{d}y\,\mathrm{d}x\\&=-4\int_{-1}^0 x\frac{y^2}{2}\Big|_0^{x+1}\mathrm{d}x=2\int_{-1}^0\left(x^3+2x^2+x\right)\mathrm{d}x\\&=2\left(\frac{1}{4}x^4+\frac{2}{3}x^3+\frac{1}{2}x^2\right)\Big|_{-1}^0=-2\left(\frac{1}{4}-\frac{2}{3}+\frac{1}{2}\right)=-\frac{1}{6}\end{aligned}$$

■

6. [2350/121923 (28 pts)] Consider the surface $z=x^2+y^2$ and vector field $\mathbf{E}=x\,\mathbf{i}+y\,\mathbf{j}+2z\left(x^2+y^2\right)\mathbf{k}$.

(a) (3 pts) Name the surface.

(b) (25 pts) Let $\mathcal{S}$ be that portion of $z=x^2+y^2$ with $0\le z\le 1$ and let $\mathcal{S}_1$ be the disk $x^2+y^2\le 1$ lying in the plane $z=1$. Then $\mathcal{S}\cup\mathcal{S}_1$ is a closed surface enclosing the solid region $\mathcal{W}$.

i. (10 pts) Compute the upward flux of $\mathbf{E}$ through $\mathcal{S}_1$.

ii. (10 pts) Compute $\iiint_{\mathcal{W}}\nabla\cdot\mathbf{E}\,\mathrm{d}V$.

iii. (5 pts) Use Gauss' Divergence Theorem to find the downward flux of $\mathbf{E}$ through $\mathcal{S}$, which is not a closed surface. Hint: no integration is necessary.

SOLUTION:

(a) circular paraboloid

(b) i. We find the upward flux of $\mathbf{E}$ through $\mathcal{S}_1$, $\iint_{\mathcal{S}_1} \mathbf{E}\cdot\mathbf{n}\,\mathrm{d}S$, by projecting $\mathcal{S}_1$, given by $g(x,y,z)=z$, onto the xy-plane.

$$g(x,y,z)=z,\ \nabla g=\mathbf{k},\ \mathcal{R}: x^2+y^2\le 1,\ \mathbf{p}=\mathbf{k},\ |\nabla g\cdot\mathbf{p}|=1$$

$$\iint_{\mathcal{S}_1} \mathbf{E}\cdot\mathbf{n}\,\mathrm{d}S=\iint_{\mathcal{R}} \mathbf{E}\cdot\frac{+\nabla g}{|\nabla g\cdot\mathbf{p}|}\,\mathrm{d}A=\iint_{\mathcal{R}} \left\langle x,y,2z\left(x^2+y^2\right)\right\rangle\cdot\frac{\langle 0,0,1\rangle}{1}\,\mathrm{d}A=\iint_{\mathcal{R}} 2z\left(x^2+y^2\right)\,\mathrm{d}A$$

(switch to polar coordinates and eliminate z using the surface $z=1$)

$$=2\int_0^{2\pi}\int_0^1 r^3\,\mathrm{d}r\,\mathrm{d}\theta=4\pi\frac{r^4}{4}\bigg|_0^1=\pi$$

ii. We have $\nabla\cdot\mathbf{E}=1+1+2\left(x^2+y^2\right)=2\left(1+x^2+y^2\right)$ so that, using cylindrical coordinates,

$$\iiint_{\mathcal{W}} 2\left(1+x^2+y^2\right)\mathrm{d}V=\int_0^{2\pi}\int_0^1\int_{r^2}^1 2\left(1+r^2\right)r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$$

$$=2\int_0^{2\pi}\int_0^1 \left(r+r^3\right)z\Big|_{r^2}^1\,\mathrm{d}r\,\mathrm{d}\theta=2\int_0^{2\pi}\int_0^1\left(r+r^3\right)\left(1-r^2\right)\mathrm{d}r\,\mathrm{d}\theta$$

$$=2\int_0^{2\pi}\int_0^1\left(r-r^5\right)\mathrm{d}r\,\mathrm{d}\theta=2\int_0^{2\pi}\left(\frac{r^2}{2}-\frac{r^6}{6}\right)\bigg|_0^1\mathrm{d}\theta=\frac{2}{3}\int_0^{2\pi}\mathrm{d}\theta=\frac{4\pi}{3}$$

iii. The downward flux through $\mathcal{S}$ and the upward flux through $\mathcal{S}_1$ correspond to the outward pointing normal of the closed surface $\mathcal{S}\cup\mathcal{S}_1$ to which Gauss' Divergence Theorem can be directly applied.

$$\iint_{\mathcal{S}\cup\mathcal{S}_1}\mathbf{E}\cdot\mathbf{n}\,\mathrm{d}S=\iint_{\mathcal{S}}\mathbf{E}\cdot\mathbf{n}\,\mathrm{d}S+\iint_{\mathcal{S}_1}\mathbf{E}\cdot\mathbf{n}\,\mathrm{d}S=\iiint_{\mathcal{W}}\nabla\cdot\mathbf{E}\,\mathrm{d}V$$

$$\iint_{\mathcal{S}}\mathbf{E}\cdot\mathbf{n}\,\mathrm{d}S=\iiint_{\mathcal{W}}\nabla\cdot\mathbf{E}\,\mathrm{d}V-\iint_{\mathcal{S}_1}\mathbf{E}\cdot\mathbf{n}\,\mathrm{d}S=\frac{4\pi}{3}-\pi=\frac{\pi}{3}$$

■