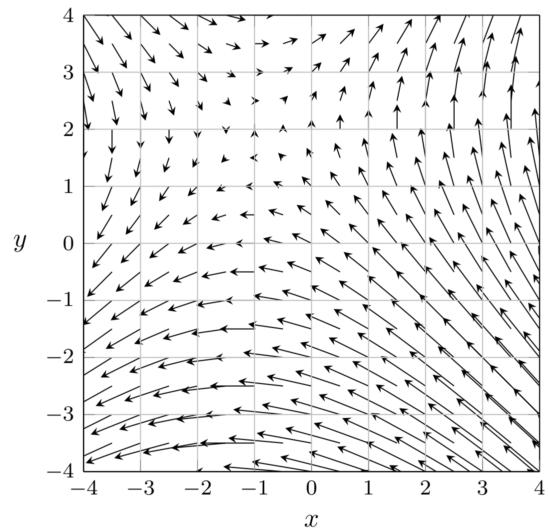
1. [2350/112923 (15 pts)] Write the word **TRUE** or **FALSE** as appropriate. Write your answers in a single column separate from any work you do. No work need be shown. No partial credit given.

   (a) The area of one side of a fence built on the curve $y = x^2$ in the $z = 0$ plane for $0 \le x \le 3$ with height $z = f(x, y) = 1 + 4y$ is $\int_0^3 (1 + 4t^2)\,\mathrm{d}t$.

   (b) The vector field $\mathbf{V} = (y - 2)\,\mathbf{i} + (x + 1)\,\mathbf{j}$ is shown in the accompanying figure.



   (c) Any vector field of the form $f(y, z)\,\mathbf{i} + g(x, z)\,\mathbf{j} + h(x, y)\,\mathbf{k}$, where the appropriate partial derivatives of f, g, h exist, is incompressible.

   (d) For any vector field $\mathbf{F}$, $\nabla \cdot (\nabla \times \mathbf{F}) = \nabla \times (\nabla \cdot \mathbf{F})$.

   (e) If a vector field $\mathbf{V}$ has only $\mathbf{i}$- and $\mathbf{k}$-components, both of which are functions of only x and z and whose partial derivatives are nonzero, then the curl of $\mathbf{V}$ will have only a $\mathbf{j}$-component.

SOLUTION:

   (a) **FALSE** The area is given by $\int_{\mathcal{C}} f(x, y)\,\mathrm{d}s$. With $\mathbf{r}(t) = \langle t, t^2 \rangle$, $\mathbf{r}'(t) = \langle 1, 2t \rangle$, $\|\mathbf{r}'(t)\| = \sqrt{1 + 4t^2}$. Then

$$\int_{\mathcal{C}} (1 + 4y)\,\mathrm{d}s = \int_0^3 f(\mathbf{r}(t))\|\mathbf{r}'(t)\|\,\mathrm{d}t = \int_0^3 \left(1 + 4t^2\right)^{3/2}\,\mathrm{d}t$$

   (b) **TRUE**

   (c) **TRUE**

$$\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial f(y, z)}{\partial x} + \frac{\partial g(x, z)}{\partial y} + \frac{\partial h(x, y)}{\partial z} = 0 + 0 + 0 = 0$$

   (d) **FALSE** $\nabla \times (\nabla \cdot \mathbf{F})$ is not defined

   (e) **TRUE** $\mathbf{V} = f(x, z)\,\mathbf{i} + g(x, z)\,\mathbf{k}$. Then

$$\operatorname{curl}\mathbf{V} = \nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f(x, z) & 0 & g(x, z) \end{vmatrix} = \left(\frac{\partial f}{\partial z} - \frac{\partial g}{\partial x}\right)\mathbf{j}$$

■

2. [2350/112923 (16 pts)] You need to compute $G = \int_{\mathcal{R}} \frac{x^2 - y^2}{x^2 + y^2}\,\mathrm{d}A$, where $\mathcal{R}$ is region in the first quadrant bounded by the curves

$$x^2 + y^2 = 4, \quad x^2 + y^2 = 8, \quad x^2 - y^2 = 2, \quad x^2 - y^2 = -2$$

(a) (2 pts) Describe in words what the quantity G represents.

(b) (14 pts) Use the change of variables $u = \frac{1}{2}\left(x^2+y^2\right)$ and $v = \frac{1}{2}\left(x^2-y^2\right)$ to set up, **but not evaluate**, an appropriate integral to compute G.

SOLUTION:

(a) The signed volume of the solid lying between the function $\dfrac{x^2-y^2}{x^2+y^2}$ and the xy-plane above/below the region $\mathcal{R}$.

(b) With $u = \frac{1}{2}\left(x^2+y^2\right)$ and $v = \frac{1}{2}\left(x^2-y^2\right)$, the new region of integration is $2 \le u \le 4$ and $-1 \le v \le 1$ and

$$u+v = x^2 \implies x = \sqrt{u+v} \text{ and } u-v = y^2 \implies y = \sqrt{u-v}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2\sqrt{u+v}} & \frac{1}{2\sqrt{u+v}} \\ \frac{1}{2\sqrt{u-v}} & -\frac{1}{2\sqrt{u-v}} \end{vmatrix} = \frac{-1}{2\sqrt{(u+v)(u-v)}} = \frac{-1}{2\sqrt{u^2-v^2}}$$

$$x^2+y^2 = u+v+u-v = 2u \text{ and } x^2-y^2 = u+v-(u-v) = 2v$$

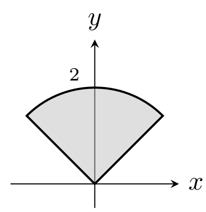
$$G = \int_{-1}^{1}\int_{2}^{4} \frac{2v}{2u}\left|\frac{-1}{2\sqrt{u^2-v^2}}\right| \mathrm{d}u\,\mathrm{d}v = \int_{-1}^{1}\int_{2}^{4} \frac{v}{2u\sqrt{u^2-v^2}}\,\mathrm{d}u\,\mathrm{d}v = \int_{2}^{4}\int_{-1}^{1} \frac{v}{2u\sqrt{u^2-v^2}}\,\mathrm{d}v\,\mathrm{d}u$$

■

3. [2350/112923 (25 pts)] Winnie the Pooh has his eyes on a honey-filled beehive in the shape of $4z = 4 - x^2 - y^2$. He can see the portion of the hive between the planes $y = x$ and $y = -x$ where $y \ge 0$. The surface of the hive is covered with bees and his friend Rabbit tells him that the density of the bees is $\delta(x,y,z) = 100y^2/\sqrt{x^2+y^2+4}$ bees per square centimeter. He also reminds him that $\sin^2 x = (1-\cos 2x)/2$. How many bees does Pooh see? (it won't be a whole number, but that's ok)

SOLUTION:

To find the number of bees, we need to compute a scalar surface integral, $\iint_{\mathcal{S}} \delta(x,y,z)\,\mathrm{d}S$, where $\mathcal{S}$ is the surface of the hive, given by $g(x,y,z) = x^2+y^2+4z = 4$. We will project the surface onto the xy-plane so that $\mathbf{p} = \mathbf{k}$. The region of integration, $\mathcal{R}$, is shown below:



Moreover,

$$\nabla g = \langle 2x, 2y, 4\rangle \implies \|\nabla g\| = \sqrt{4x^2+4y^2+16} = 2\sqrt{x^2+y^2+4} \text{ and } |\nabla g \cdot \mathbf{p}| = 4$$

Thus,

$$\text{Number of bees} = \iint_{\mathcal{S}} \frac{100y^2}{\sqrt{x^2+y^2+4}}\,\mathrm{d}S = \iint_{\mathcal{R}} \frac{100y^2}{\sqrt{x^2+y^2+4}}\left(\frac{2\sqrt{x^2+y^2+4}}{4}\right)\mathrm{d}A = 50\iint_{\mathcal{R}} y^2\,\mathrm{d}A$$

This integral is an obvious candidate for polar coordinates. Making this transformation gives

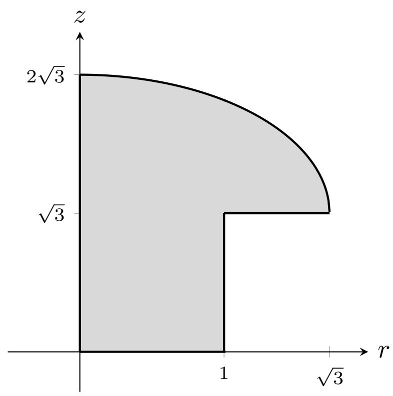
$$\begin{aligned} 50\iint_{\mathcal{R}} y^2\,\mathrm{d}A &= 50\int_{\pi/4}^{3\pi/4}\int_0^2 r^3\sin^2\theta\,\mathrm{d}r\,\mathrm{d}\theta \\ &= 50\int_{\pi/4}^{3\pi/4}\left(\frac{1-\cos 2\theta}{2}\right)\mathrm{d}\theta\left(\int_0^2 r^3\,\mathrm{d}r\right) \\ &= 25\left(\theta - \frac{1}{2}\sin 2\theta\right)\Bigg|_{\pi/4}^{3\pi/4}\left(\frac{r^4}{4}\right)\Bigg|_0^2 \\ &= 100\left(\frac{\pi}{2}+1\right) = 50\pi + 100 \text{ bees} \end{aligned}$$

Had the conversion to polar coordinates not been made, much more effort is required to evaluate the following:

$$\text{Number of bees} = \int_{-\sqrt{2}}^{0}\int_{-x}^{\sqrt{4-x^2}} y^2 \,\mathrm{d}y\,\mathrm{d}x + \int_{0}^{\sqrt{2}}\int_{x}^{\sqrt{4-x^2}} y^2 \,\mathrm{d}y\,\mathrm{d}x$$

■

4. [2350/112923 (24 pts)] A constant θ slice through a delicious red velvet cupcake is shown in the following figure. The top of the cupcake is a portion of $x^2+y^2+\left(z-\sqrt{3}\right)^2 = 3$ and the mass density of the cupcake is $\delta(x,y,z) = y^2/(z+1)$. Set up, **do not evaluate**, integral(s) to compute the following, using the given integration order.



(a) (4 pts) The mass of the portion of the cupcake above the plane $z=\sqrt{3}$, $\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$.

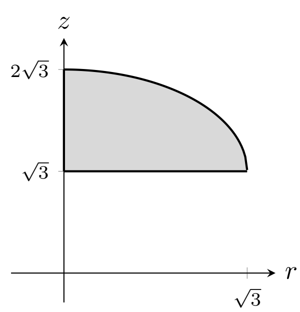
(b) (8 pts) The mass of the entire cupcake, $\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$.

(c) (12 pts) The mass of the entire cupcake, $\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$.

<u>**SOLUTION:**</u>

(a)

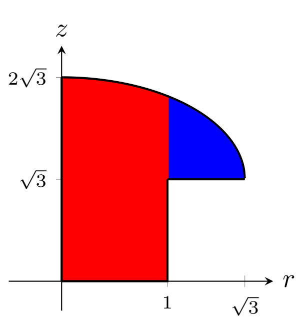
$$\text{Mass} = \int_{-\sqrt{3}}^{\sqrt{3}}\int_{-\sqrt{3-y^2}}^{\sqrt{3-y^2}}\int_{\sqrt{3}}^{\sqrt{3}+\sqrt{3-x^2-y^2}} \frac{y^2}{z+1}\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$$

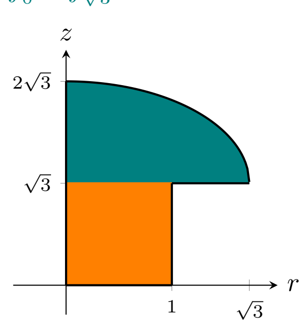


(b) Two alternatives.

$$\text{Mass} = \int_0^{2\pi}\int_0^{1}\int_0^{\sqrt{3}+\sqrt{3-r^2}} \frac{r^3\sin^2\theta}{z+1}\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta + \int_0^{2\pi}\int_1^{\sqrt{3}}\int_{\sqrt{3}}^{\sqrt{3}+\sqrt{3-r^2}} \frac{r^3\sin^2\theta}{z+1}\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$$

$$\text{Mass} = \int_0^{2\pi}\int_0^{1}\int_0^{\sqrt{3}} \frac{r^3\sin^2\theta}{z+1}\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta + \int_0^{2\pi}\int_0^{\sqrt{3}}\int_{\sqrt{3}}^{\sqrt{3}+\sqrt{3-r^2}} \frac{r^3\sin^2\theta}{z+1}\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$$

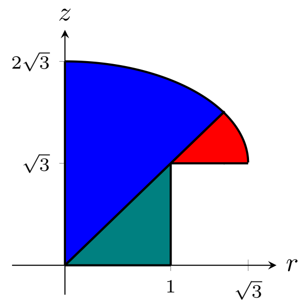




(c)

$$\text{Mass} = \int_0^{2\pi}\int_0^{\pi/6}\int_0^{2\sqrt{3}\cos\phi} \frac{\rho^4\sin^3\phi\sin^2\theta}{\rho\cos\phi+1}\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta + \int_0^{2\pi}\int_{\pi/6}^{\pi/4}\int_{\sqrt{3}\sec\phi}^{2\sqrt{3}\cos\phi} \frac{\rho^4\sin^3\phi\sin^2\theta}{\rho\cos\phi+1}\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$$

$$+\int_0^{2\pi}\int_{\pi/6}^{\pi/2}\int_0^{\csc\phi} \frac{\rho^4\sin^3\phi\sin^2\theta}{\rho\cos\phi+1}\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$$



5. [2350/112923 (20 pts)] A thin triangular plate (lamina) lying in the xy-plane with density $\rho(x,y) = 2y$ is bounded by the lines $y = 1$, $y = 2 - x$ and $y = 2 + x$. The mass of the lamina is $m = 8/3$.

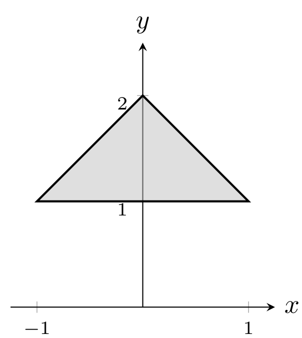
(a) (8 pts) Find the moment of inertia about the x-axis, integrating with respect to x first.

(b) (8 pts) Find the moment of inertia about the y-axis, integrating with respect to y first.

(c) (4 pts) Find the radii of gyration with respect to the x- and y-axes.

SOLUTION:

Sketch of the plate.



(a)

$$I_x = \iint\limits_{\text{lamina}} y^2\rho(x,y)\,\mathrm{d}A = \int_1^2\int_{y-2}^{2-y} 2y^3\,\mathrm{d}x\,\mathrm{d}y$$

$$= 2\int_1^2 y^3x\Big|_{y-2}^{2-y}\,\mathrm{d}y = 2\int_1^2\left(4y^3-2y^4\right)\,\mathrm{d}y$$

$$= 2\left(y^4-\frac{2}{5}y^5\right)\Big|_1^2 = \frac{26}{5}$$

(b) We can exploit symmetry across the y-axis here since the density is independent of x.

$$I_y = \iint\limits_{\text{lamina}} x^2\rho(x,y)\,\mathrm{d}A = \int_{-1}^0\int_1^{2+x} 2x^2y\,\mathrm{d}y\,\mathrm{d}x + \int_0^1\int_1^{2-x} 2x^2y\,\mathrm{d}y\,\mathrm{d}x = 2\int_0^1\int_1^{2-x} 2x^2y\,\mathrm{d}y\,\mathrm{d}x$$

$$= 2\int_0^1 x^2y^2\Big|_1^{2-x}\,\mathrm{d}x = 2\int_0^1 x^2\left[(2-x)^2-1\right]\,\mathrm{d}x = 2\int_0^1\left(x^4-4x^3+3x^2\right)\mathrm{d}x$$

$$= 2\left(\frac{x^5}{5}-x^4+x^3\right)\Big|_0^1 = \frac{2}{5}$$

(c)

$$\text{radius of gyration with respect to the } x\text{-axis : } \bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{26/5}{8/3}} = \sqrt{\frac{39}{20}} = \frac{1}{2}\sqrt{\frac{39}{5}} = \frac{\sqrt{195}}{10}$$

$$\text{radius of gyration with respect to the } y\text{-axis : } \bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{2/5}{8/3}} = \sqrt{\frac{3}{20}} = \frac{1}{2}\sqrt{\frac{3}{5}} = \frac{\sqrt{15}}{10}$$

■