1. [2350/102523 (15 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

   (a) The function $g(x,y) = \begin{cases} \dfrac{1+xy}{y^2-x^2} & (x,y) \neq (-1,1) \\ 0 & (x,y) = (-1,1) \end{cases}$ is continuous at $(x,y) = (-1,1)$.

   (b) The function $f(x,y) = e^{-x-y}\cos\left(\pi x^2 y^3\right)$ is guaranteed to have a maximum and minimum value on the triangular region bounded by $x \geq 0, y \geq 0, x+y < 1$.

   (c) If $x(u,v) = 6\left(\sqrt{u}+1\right) - v^2, u = rst$ and $v = \dfrac{rs}{t}$, $\dfrac{\partial x}{\partial t} = 3\sqrt{\dfrac{rs}{t}} + 2\dfrac{r^2s^2}{t^3}$.

   (d) If $h(x,y)$ is a function whose partial derivatives of all orders are continuous throughout $\mathbb{R}^2$ and if $h_x(2,3) = h_y(2,3) = 0$, then $h(2,3)$ must be a local extreme value of $h(x,y)$.

   (e) The equation of the tangent plane to the surface $(x-1)^2 + (y+2)^2 - z^2 = -16$ at the point $(-1,2,6)$ is $x - 2y + 3z = 13$.

   SOLUTION:

   (a) **FALSE** Although $g(-1,1)$ is defined, $\displaystyle\lim_{(x,y)\to(-1,1)} g(x,y)$ does not exist. Thus $g(x,y)$ is not continuous at $(-1,1)$. Note that direct substitution yields the indeterminate form $0/0$. If we approach $(-1,1)$ along the line $y=1$ we have

   $$\lim_{(x,1)\to(-1,1)} \frac{1+x}{1-x^2} = \lim_{(x,1)\to(-1,1)} \frac{1+x}{(1+x)(1-x)} = \lim_{(x,1)\to(-1,1)} \frac{1}{1-x} = \frac{1}{2}$$

   If we approach $(-1,1)$ along the line $x=-1$ we have

   $$\lim_{(-1,y)\to(-1,1)} \frac{1-y}{y^2-1} = \lim_{(-1,y)\to(-1,1)} \frac{1-y}{(y+1)(y-1)} = \lim_{(-1,y)\to(-1,1)} \frac{-1}{y+1} = -\frac{1}{2}$$

   This proves that the limit fails to exist.

   (b) **FALSE** The function is continuous throughout $\mathbb{R}^2$ and thus on the triangular region. However, the triangular region is bounded, but not closed since the hypotenuse is not included in the region. Therefore, the Extreme Value Theorem does not apply and we are guaranteed nothing.

   (c) **TRUE**

   $$\frac{\partial x}{\partial t} = \frac{\partial x}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial t} = \frac{3}{\sqrt{u}}(rs) - 2v\left(-\frac{rs}{t^2}\right) = \frac{3rs}{\sqrt{rst}} - 2\left(\frac{rs}{t}\right)\left(-\frac{rs}{t^2}\right) = 3\sqrt{\frac{rs}{t}} + 2\frac{r^2s^2}{t^3}$$

   (d) **FALSE** $h(2,3)$ could be a saddle point.

   (e) **TRUE** With $F(x,y,z) = (x-1)^2 + (y+2)^2 - z^2$, the normal to the surface is $\nabla F(x,y,z) = \langle 2(x-1), 2(y+2), -2z\rangle \implies \nabla F(-1,2,6) = \langle -4, 8, -12\rangle$. Then

   $$-4(x+1) + 8(y-2) - 12(z-6) = 0 \implies -4x + 8y - 12z = -52 \implies x - 2y + 3z = 13$$

   ■

2. [2350/102523 (17 pts)] You are wandering around on the surface $xy - z^2 = 1$. Use Lagrange Multipliers to determine the closest that you will get to the origin.

   SOLUTION:

   We need to minimize the distance from the surface to the origin. To simplify things, we will use the distance squared, minimizing $f(x,y,z) = x^2 + y^2 + z^2$ subject to the constraint $g(x,y,z) = xy - z^2 = 1$.

   $$2x = \lambda y \tag{1}$$
   $$2y = \lambda x \tag{2}$$
   $$2z = \lambda(-2z) \tag{3}$$
   $$xy - z^2 = 1 \tag{4}$$

Equation (3) can be written as $z(1+\lambda)=0$. If $z=0$, Eq. (4) gives $xy=1 \implies y=1/x$. Then Eq. (1) is $2x=\lambda/x$ or $\lambda=2x^2$ and Eq. (2) is $2/x=\lambda x$ or $\lambda=2/x^2$ which, when equated, give $x^4=1 \implies x=\pm 1 \implies y=\pm 1$. If $z\neq 0$, then $\lambda=-1$ and Eq. (1) is $2x=-y$ and Eq. (2) is $2y=-x$ the only solution of which is $x=y=0$. But in that case, Eq. (4) has no real solution. The critical points are thus $(1,1,0)$ and $(-1,-1,0)$. The distance to the origin in these cases is $\sqrt{2}$. Since the constraint is unbounded, f has no maximum on the constraint surface so this distance must be a minimum. ■

3. [2350/102523 (17 pts)] You are told that for a given function $f(x,y)$, its gradient is $\nabla f(x,y)=\left(3x^2+3y\right)\mathbf{i}+\left(3y^2+3x\right)\mathbf{j}$. Find and classify the critical points of $f(x,y)$.

SOLUTION:

Since the gradient is defined throughout $\mathbb{R}^2$, critical points occur where the gradient is the zero vector.

$$f_x = 3x^2+3y=0 \tag{5}$$

$$f_y = 3y^2+3x=0 \tag{6}$$

From Eq. (5), $x^2=-y \implies x^4=y^2$ which, when substituted into Eq. (6), yields $x^4+x=x(x^3+1)=0 \implies x=0,-1$. From this then, $y=0,-1$. The critical points are $(0,0)$ and $(-1,-1)$. To classify them, $f_{xx}=6x$, $f_{yy}=6y$, $f_{xy}=3$ and

$$D(x,y)=(6x)(6y)-3^2=36xy-9$$

$$D(0,0)=-9<0 \implies (0,0) \text{ is a saddle point}$$

$$D(-1,-1)=36-9=27>0 \text{ and } f_{xx}(-1,-1)=-6<0 \implies f(-1,-1) \text{ is a local maximum}$$

■

4. [2350/102523 (31 pts)] Winnie the Pooh is on another quest for honey in the Hundred Acre Wood. The elevation of the ground in the woods is given by the function $h(x,y)=\ln xy$.

   (a) (4 pts) Find the domain of the elevation function, $h(x,y)$.

   (b) (7 pts) Find the slope of the ground in the direction of $2\,\mathbf{i}+\mathbf{j}$ at the point $(1,1,0)$.

   (c) (10 pts) Pooh is walking towards a beehive full of honey, following the path in the xy-plane given by $\mathbf{r}(t)=e^t\,\mathbf{i}+e^{2t}\,\mathbf{j}$. As he passes through the point $\left(e,e^2\right)$, how fast is his elevation changing with respect to time? What is his elevation there?

   (d) (10 pts) Pooh's friend Piglet is standing guard at the honey-filled beehive located at the point $(3,9,3\ln 3)$. Those pesky bees note that something is amiss with their honey and come after Piglet. In what direction should Piglet start running in order to begin losing elevation the fastest and escape the bees? Write your answer as a vector in the xy-plane. What is Piglet's instantaneous rate of change of elevation?

SOLUTION:

(a) $\left\{(x,y)\in\mathbb{R}^2 \,\middle|\, xy>0\right\}$ (Quadrant I and III)

(b)

$$\nabla h(x,y)=\left\langle \frac{1}{x},\frac{1}{y}\right\rangle \implies \nabla h(1,1)=\langle 1,1\rangle$$

$$\left.\frac{\mathrm{d}h}{\mathrm{d}s}\right|_{(1,1)}=D_{\mathbf{u}}h(1,1)=\nabla h(1,1)\cdot\left\langle \frac{2}{\sqrt{5}},\frac{1}{\sqrt{5}}\right\rangle=\langle 1,1\rangle\cdot\left\langle \frac{2}{\sqrt{5}},\frac{1}{\sqrt{5}}\right\rangle=\frac{3}{\sqrt{5}}=\frac{3\sqrt{5}}{5}$$

(c) Pooh gets to the point $\left(e,e^2\right)$ when $t=1$. We have $\mathbf{r}'(t)=\left\langle e^t,2e^{2t}\right\rangle$.

$$\left.\frac{\mathrm{d}h}{\mathrm{d}t}\right|_{t=1}=\nabla h\left(e,e^2\right)\cdot\mathbf{r}'(1)=\left\langle \frac{1}{e},\frac{1}{e^2}\right\rangle\cdot\left\langle e,2e^2\right\rangle=3$$

When $t=1$, $\mathbf{r}(1)=e\,\mathbf{i}+e^2\,\mathbf{j} \implies x=e, y=e^2$, so that Pooh's elevation is $h\left(e,e^2\right)=\ln\left(e\cdot e^2\right)=\ln e^3=3$.

(d) Piglet needs to run in the direction of $-\nabla h(3,9)=-\dfrac{1}{3}\,\mathbf{i}-\dfrac{1}{9}\,\mathbf{j}$. His instantaneous rate of descent is

$$-\|-\nabla h(3,9)\|=-\sqrt{\frac{1}{9}+\frac{1}{81}}=-\frac{\sqrt{10}}{9}$$

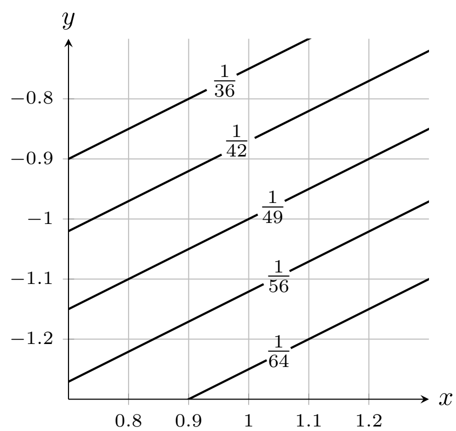
5. [2350/102523 (20 pts)] The following problems are not related.

(a) (10 pts) According to the ideal gas law, the pressure (P), temperature (T) and volume (V) of a confined gas are related by $P = RT/V$ where R is a constant. Use differentials to approximate the percentage change in pressure $(\mathrm{d}P/P)$ if the temperature of a gas is increased 3% and the volume is increased by 5%.

(b) (10 pts) Your friends have found the first order Taylor polynomial for the function $f(x, y)$, centered at $(1, -1)$. They want to use this polynomial to approximate $f(x, y)$ when x and y satisfy the inequalities $|x - 1| \le 0.1$ and $|y + 1| \le 0.2$ and need to know about the error in the approximation. They have been kind enough to provide you with the following derivative information,

$$f_{xx}(x,y) = \frac{-4}{(1+2x-4y)^2}, \quad f_{xy}(x,y) = \frac{8}{(1+2x-4y)^2}, \quad f_{yy}(x,y) = \frac{-16}{(1+2x-4y)^2},$$

as well as a graph of the level curves of $1/(1+2x-4y)^2$ shown in the figure. Based on the given information, what is the largest error your friends can expect the approximation to contain?



SOLUTION:

(a) Note that P is a function of T and V, that is, $P(T, V)$. Thus

$$\mathrm{d}P = \frac{\partial P}{\partial T}\,\mathrm{d}T + \frac{\partial P}{\partial V}\,\mathrm{d}V = \frac{R}{V}\,\mathrm{d}T - \frac{RT}{V^2}\,\mathrm{d}V$$
$$\frac{\mathrm{d}P}{P} = \frac{R}{PV}\,\mathrm{d}T - \frac{RT}{V^2 P}\,\mathrm{d}V$$
$$\frac{\mathrm{d}P}{P} = \frac{\mathrm{d}T}{T} - \frac{\mathrm{d}V}{V}$$

If $\mathrm{d}T/T = 0.03$ and $\mathrm{d}V/V = 0.05$ then $\mathrm{d}P/P = 0.03 - 0.05 = -0.02$. Thus, there is about a 2% decrease in pressure.

(b) The relevant error formula is $|E(x,y)| \le \frac{M}{2!}\left(|x-a| + |y-b|\right)^2$. To use the error formula, $a = 1, b = -1$, and we need to find an upper bound, M, on the absolute value of the second derivatives of $f(x, y)$ over the region of interest, $|x - 1| \le 0.1$ and $|y + 1| \le 0.2$. The region can be described as $0.9 \le x \le 1.1$ and $-1.2 \le y \le -0.8$. From the level curves in the figure, $\frac{1}{(1+2x-4y)^2} \le \frac{1}{36}$ in the region of interest. Thus

$$|f_{xx}| = \frac{4}{(1+2x-4y)^2} \le \frac{4}{36} = \frac{1}{9}, \quad |f_{xy}| = \frac{8}{(1+2x-4y)^2} \le \frac{8}{36} = \frac{2}{9}, \quad |f_{yy}| = \frac{16}{(1+2x-4y)^2} \le \frac{16}{36} = \frac{4}{9}$$

so we choose $M = \frac{4}{9}$ and

$$|E(x,y)| \le \frac{\frac{4}{9}}{2}\left(0.1 + 0.2\right)^2 = \frac{2}{9}\left(\frac{9}{100}\right) = \frac{1}{50}$$

You can tell your friends that the maximum error they should expect in their approximation to be $\frac{1}{50}$.