1. [2350/092723 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

   (a) If $\mathbf{a}$ is a unit vector and $\mathbf{b} = 3\mathbf{a}$, then $\mathbf{a} \cdot \mathbf{b} = 0$ and $\mathbf{b} \times \mathbf{a} = 3\mathbf{b}$.

   (b) For any smooth path $\mathbf{r}(t)$, the magnitude of $\mathbf{T} \times \mathbf{B} \times \mathbf{N}$ is zero.

   (c) Consider the arbitrary, non-zero, vectors $\mathbf{U}$ and $\mathbf{V}$. The projection of $\mathbf{U} \times \mathbf{V}$ onto $\mathbf{U}$ is the zero vector, $\mathbf{0}$.

   (d) The work done moving 6 units in the direction of $\mathbf{a} = 6\,\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}$ subject to the force $\mathbf{F} = 2\,\mathbf{i} + 7\mathbf{j} + 3\,\mathbf{k}$ Nt is 35 Nt-m.

   (e) Suppose $\mathbf{A}$ and $\mathbf{B}$ are arbitrary, nonzero vectors with equal magnitude. Then $\mathbf{A} + \mathbf{B}$ and $\mathbf{A} - \mathbf{B}$ are orthogonal.

   SOLUTION:

   (a) **FALSE** $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot 3\,\mathbf{a} = 3\mathbf{a} \cdot \mathbf{a} = 3$ and $\mathbf{b} \times \mathbf{a} = 3\,\mathbf{a} \times \mathbf{a} = \mathbf{0}$

   (b) **TRUE** $\mathbf{T} \times \mathbf{B} \times \mathbf{N} = \mathbf{T} \times (\mathbf{B} \times \mathbf{N}) = \mathbf{T} \times (-\mathbf{T}) = -\mathbf{T} \times \mathbf{T} = \mathbf{0}$ and $\|\mathbf{0}\| = 0$; Alternatively, $(\mathbf{T} \times \mathbf{B}) \times \mathbf{N} = -\mathbf{N} \times \mathbf{N} = \mathbf{0}$

   (c) **TRUE** $\text{proj}_{\mathbf{U}} \mathbf{U} \times \mathbf{V} = \left[\dfrac{\mathbf{U} \cdot (\mathbf{U} \times \mathbf{V})}{\mathbf{U} \cdot \mathbf{U}} \right] \mathbf{U} = \mathbf{0}$ since $\mathbf{U}$ is orthogonal to $\mathbf{U} \times \mathbf{V}$

   (d) **FALSE** Work $= \mathbf{F} \cdot \mathbf{D} = \langle 2, 7, 3 \rangle \cdot 6 \dfrac{\mathbf{a}}{\|\mathbf{a}\|} = 6\langle 2, 7, 3 \rangle \cdot \left\langle \dfrac{6}{7}, \dfrac{2}{7}, \dfrac{3}{7} \right\rangle = 30$ Nt-m

   (e) **TRUE** $(\mathbf{A} + \mathbf{B}) \cdot (\mathbf{A} - \mathbf{B}) = \mathbf{A} \cdot \mathbf{A} - \mathbf{A} \cdot \mathbf{B} + \mathbf{B} \cdot \mathbf{A} - \mathbf{B} \cdot \mathbf{B} = \|\mathbf{A}\|^2 - \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{B} - \|\mathbf{B}\|^2 = \|\mathbf{A}\|^2 - \|\mathbf{B}\|^2 = 0$ since $\|\mathbf{A}\| = \|\mathbf{B}\|$

2. [2350/092723 (30 pts)] Winnie the Pooh is on an adventure to track down some of his beloved honey. He spies a beehive high up in a tree and his friend Christopher Robin has given him some balloons that will lift him up to the hive. When $t = 0$, Christopher Robin releases Pooh Bear from the point $(0, 1, 0)$ and he floats along the path $\mathbf{r}(t) = \sin 4t\,\mathbf{i} + \cos 4t\,\mathbf{j} + 2t\,\mathbf{k}$ until $t = 2\pi$, at which time he arrives at the beehive.

   (a) (8 pts) How far did Pooh travel?

   (b) (6 pts) To get a better view of Pooh when he is at the beehive enjoying the honey, Christopher walks into a small pit located at $(5, 6, -\pi)$. How far is he from the beehive?

   (c) (8 pts) As Pooh is indulging himself, a gust of wind in the direction $3\,\mathbf{i} - 2\,\mathbf{j} + \mathbf{k}$ hits when $u = 0$ and blows him from the beehive in a straight line for $u > 0$. Find the parametric equations of this line, using u as the parameter.

   (d) (8 pts) A swarm of bees, to whom the honey belongs, is flying along the path $\mathbf{r}_b(u) = (6 + u)\,\mathbf{i} + \left(4 - u^2\right)\mathbf{j} + \left(\sqrt{3u} + 4\pi\right)\mathbf{k}$. They are irritated that Pooh has stolen their honey and are out to get him. Where do the bees meet up with Pooh? [Note that this is the same u as in part (c)].

   SOLUTION:

   (a)

$$\mathrm{r}'(t) = 4\cos 4t\,\mathbf{i} - 4\sin 4t\,\mathbf{j} + 2\,\mathrm{k}$$

$$\|\mathbf{r}'(t)\| = \sqrt{(4\cos 4t)^2 + (-4\sin 4t)^2 + 2^2} = \sqrt{20} = 2\sqrt{5}$$

$$s = \int_0^{2\pi} 2\sqrt{5}\,\mathrm{d}t = 4\pi\sqrt{5}$$

   (b) The beehive is at the point $(0, 1, 4\pi)$ so the distance is

$$d = \sqrt{(0-5)^2 + (1-6)^2 + (4\pi + \pi)^2} = \sqrt{50 + 25\pi^2} = \sqrt{25\left(2 + \pi^2\right)} = 5\sqrt{2 + \pi^2}$$

   (c) The point on the line is $(0, 1, 4\pi)$ and the line's direction is $\langle 3, -2, 1 \rangle$. Thus the vector equation of the line is $\mathbf{L}(t) = \langle 0, 1, 4\pi \rangle + u\langle 3, -2, 1 \rangle$ with parametric equations

$$x(u) = 3u \qquad y(u) = 1 - 2u \qquad z(u) = 4\pi + u$$

(d) We are told that the bees meet up with Pooh so we need to find the u that makes the x-, y- and z-coordinates of the line along which Pooh is moving and the bee's path the same. That is,

$$3u = 6 + u \quad (1)$$

$$1 - 2u = 4 - u^2 \quad (2)$$

$$4\pi + u = \sqrt{3u} + 4\pi \quad (3)$$

Eq. (1) requires $u = 3$ which also satisfies Eq. (2) and (3). The bees meet Pooh at the point $(9, -5, 4\pi + 3)$.

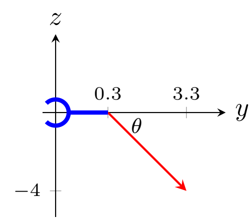
3. [2350/092723 (17 pts)] The following problems are not related.

(a) (8 pts) A wrench 30 cm long lies along the positive y-axis and grips a bolt at the origin. A force is applied in the direction of $\langle 0, 3, -4\rangle$ at the end of the wrench. Find the magnitude of the force needed to supply 100 Nt-m of torque to the bolt.

(b) (9 pts) Consider the equation $-x^2 + 4x + 3z^2 + ay^2 + 2y + 2 = 0$. Determine the quadric surface that results when $a = -1, 0, 1$.

SOLUTION:

(a) Sketch of the situation. Note that θ is one of the angles of a 3-4-5 right triangle and the proper units for the wrench (shown in blue) are meters.



$$\|\boldsymbol{\tau}\| = \|\mathbf{r} \times \mathbf{F}\| = \|\mathbf{r}\|\|\mathbf{F}\| \sin\theta \implies \|\mathbf{F}\| = \frac{\|\boldsymbol{\tau}\|}{\|\mathbf{r}\| \sin\theta} = \frac{100 \text{ Nt-m}}{(0.3 \text{ m})\,(4/5)} = \frac{1250}{3} \text{ Nt}$$

(b) Begin by completing the square in x giving $-(x-2)^2+3z^2+ay^2+2y+6 = 0$. Then if $a = 0$ we have $y = \frac{1}{2}(x-2)^2 - \frac{3}{2}z^2 - 3$, which is a hyperbolic paraboloid. If $a \neq 0$, complete the square in y, yielding

$$-(x-2)^2 + 3z^2 + a\left(y^2 + \frac{2}{a} + \frac{1}{a^2} - \frac{1}{a^2}\right) + 6 = 0$$

$$-(x-2)^2 + 3z^2 + a\left(y + \frac{1}{a}\right)^2 = \frac{1}{a} - 6$$

$$a = 1: \quad -(x-2)^2 + 3z^2 + (y+1)^2 = -5 \implies \text{hyperboloid of two sheets}$$

$$a = -1: \quad -(x-2)^2 + 3z^2 - (y-1)^2 = -7 \implies (x-2)^2 - 3z^2 + (y-1)^2 = 7 \implies \text{hyperboloid of one sheet}$$

4. [2350/092723 (17 pts)] The following problems are not related.

(a) (9 pts) Consider the vectors $\mathbf{u} = 2\,\mathbf{i} - 3\,\mathbf{j}$, $\mathbf{v} = t\,\mathbf{i} + 3\,\mathbf{k}$, and $\mathbf{w} = -3\,\mathbf{i} + 2\,\mathbf{j} + t\,\mathbf{k}$. For what value(s) of t will the parallelepiped formed by the three vectors have a volume of 24 units?

(b) (8 pts) A 100-meter dash is run on a track in the direction of the vector $\mathbf{a} = 5\,\mathbf{i} + 12\,\mathbf{j}$. The wind velocity $\mathbf{w} = 7\,\mathbf{i} + 2\,\mathbf{j}$ km/hr. The rules say that the wind speed in the direction of the race must not exceed 5 km/hr. Will the race results be disqualified due to an excessive wind? Justify your answer using Calculus 3 concepts.

SOLUTION:

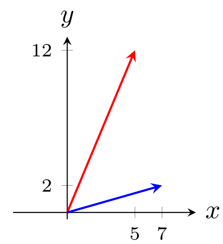
(a) The volume of the parallelepiped is given by the absolute value of the scalar triple product of the three vectors

$$\text{Volume} = |(2\,\mathbf{i} - 3\,\mathbf{j}) \cdot [(t\,\mathbf{i} + 3\,\mathbf{k}) \times (-3\,\mathbf{i} + 2\,\mathbf{j} + t\,\mathbf{k})]| = \left| \begin{vmatrix} 2 & -3 & 0 \\ t & 0 & 3 \\ -3 & 2 & t \end{vmatrix} \right| = \left|3t^2 + 15\right| = 3t^2 + 15$$

$$3t^2 + 15 = 24 \implies 3t^2 = 9 \implies t = \pm\sqrt{3}$$

We were able to eliminate the absolute value since $3t^2 + 15 > 0$ for all t.

(b) Sketch. The track is represented by the red vector and the wind by the blue vector.



The wind speed in the direction of the track is the length of the projection of the wind vector onto the vector representing the track, or the component of $\mathbf{w}$ in the direction of $\mathbf{a}$. That is

$$\text{comp}_{\mathbf{a}}\mathbf{w} = \frac{\mathbf{a}\cdot\mathbf{w}}{\|\mathbf{a}\|} = \frac{\langle 5, 12\rangle \cdot \langle 7, 2\rangle}{\sqrt{5^2+12^2}} = \frac{59}{13} < \frac{65}{13} = 5$$

The race results will be valid since the wind speed in the direction of the track is less than 5 km/hr.

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5. [2350/092723 (26 pts)] A comet is flying through space along the path given by $\mathbf{r}(t) = \left(\frac{t^2}{2} + t\right)\mathbf{i} + \frac{t^2}{2}\mathbf{j} + (t-1)\,\mathbf{k}$ where t (time) is a real number. Hint: You do not have to find $\mathbf{T}$ or $\mathbf{N}$ to do this problem.

(a) (8 pts) Find the point on the path where the comet's speed is not changing.

(b) (8 pts) Is the direction of the comet always changing? Justify your answer mathematically.

(c) (10 pts) Find the equation of the normal plane (formed by the normal and binormal vectors) when the normal plane is parallel to the plane $x + 2y - z = 1$. Write your answer in the form $ax + by + cz = d$.

SOLUTION:

(a) Alternative 1:

$$\mathbf{v}(t) = \mathbf{r}'(t) = (t+1)\,\mathbf{i} + t\,\mathbf{j} + \mathbf{k}$$

$$\|\mathbf{v}(t)\| = \sqrt{(t+1)^2 + t^2 + 1} = \sqrt{2\left(t^2+t+1\right)}$$

$$\frac{\mathrm{d}\|\mathbf{v}(t)\|}{\mathrm{d}t} = \frac{2(2t+1)}{2\sqrt{2\left(t^2+t+1\right)}} = \frac{2t+1}{\sqrt{2\left(t^2+t+1\right)}}$$

This latter expression vanishes when $t = -\frac{1}{2}$ which is the time when the speed of the comet is not changing. This happens at the point $\mathbf{r}\left(-\frac{1}{2}\right) = -\frac{3}{8}\mathbf{i} + \frac{1}{8}\mathbf{j} - \frac{3}{2}\mathbf{k}$

Alternative 2:

$$\mathbf{r}'(t) = (t+1)\,\mathbf{i} + t\,\mathbf{j} + \mathbf{k}$$

$$\mathbf{r}''(t) = \mathbf{i} + \mathbf{j}$$

$$a_T(t) = \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{\|\mathbf{r}'(t)\|} = \frac{\left[(t+1)\,\mathbf{i} + t\,\mathbf{j} + \mathbf{k}\right]\cdot(\mathbf{i}+\mathbf{j})}{\sqrt{2\left(t^2+t+1\right)}} = \frac{2t+1}{\sqrt{2\left(t^2+t+1\right)}}$$

The speed of the comet is not changing when the tangential acceleration is zero. This occurs when $t = -\frac{1}{2}$ which happens at the point $\mathbf{r}\left(-\frac{1}{2}\right) = -\frac{3}{8}\mathbf{i} + \frac{1}{8}\mathbf{j} - \frac{3}{2}\mathbf{k}$.

(b) Yes, the comet's direction is always changing.

Alternative 1: Show that the normal component of the acceleration is never 0.

$$a_N(t) = \frac{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\|\left[(t+1)\,\mathbf{i} + t\,\mathbf{j} + \mathbf{k}\right]\times(\mathbf{i}+\mathbf{j})\|}{\sqrt{2\left(t^2+t+1\right)}} = \frac{\|-\mathbf{i}+\mathbf{j}+\mathbf{k}\|}{\sqrt{2\left(t^2+t+1\right)}} = \frac{\sqrt{3}}{\sqrt{2\left(t^2+t+1\right)}}$$

Alternative 2: Show that the curvature is never 0.

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\|\left[(t+1)\,\mathbf{i} + t\,\mathbf{j} + \mathbf{k}\right] \times (\mathbf{i} + \mathbf{j})\,\|}{\left[\sqrt{2\,(t^2+t+1)}\right]^3} = \frac{\|-\mathbf{i} + \mathbf{j} + \mathbf{k}\|}{\left[2\,(t^2+t+1)\right]^{3/2}} = \frac{\sqrt{3}}{\left[2\,(t^2+t+1)\right]^{3/2}}$$

(c) A normal to the normal plane of the path is the tangent vector, $\mathbf{r}'(t) = (t+1)\,\mathbf{i} + t\,\mathbf{j} + \mathbf{k}$. We need this to be parallel to (equivalently, a scalar multiple of) the normal to the given plane, which is $\mathbf{i} + 2\,\mathbf{j} - \mathbf{k}$. That is, for some scalar c

$$\langle t+1, t, 1\rangle = c\langle 1, 2, -1\rangle$$

From the $\mathbf{k}$ component we see that $c = -1$ and the other components give

$$t + 1 = -1 \implies t = -2$$

$$t = -2$$

We could have found t by determining where $\mathbf{r}'(t) \times (\mathbf{i} + 2\,\mathbf{j} - \mathbf{k}) = \mathbf{0}$. A normal vector to the plane we seek is thus $\mathbf{r}'(-2) = \langle -1, -2, 1\rangle$ and a point in the plane is $\mathbf{r}(-2) = \langle 0, 2, -3\rangle$ so that

$$-1(x-0) - 2(y-2) + 1(z+3) = 0 \implies x + 2y - z = 7$$

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