1. [2350/121222 (21 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

   (a) The error in the quadratic Taylor polynomial for a function $g(x,y)$ is related to g_{xx}, g_{xy} and g_{yy}.

   (b) The traces in all planes parallel to all coordinate planes of the function $2z + 5x^2 - 3y^2 = 0$ are parabolas.

   (c) The principal unit normal vector, $\mathbf{N}$, for the helical path $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$, $t \geq 0$ is always parallel to the xy-plane.

   (d) The line of intersection of the planes $\frac{1}{2}x + y + 4z = 10$ and $x + 2y + 8z = -10$ has direction vector $\langle 8, 4, 1\rangle$.

   (e) The three nonzero vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, all lie in the same plane if $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$.

   (f) The Cartesian/rectangular ($z = -\sqrt{x^2 + y^2}$), cylindrical ($z = -r$) and spherical ($\phi = 3\pi/4$) equations represent the same surface.

   (g) An object moving with velocity $\mathbf{v}$ on a circle of radius 4 experiences a larger normal component of acceleration than one traveling at the same velocity on a circle of radius 1/4.

SOLUTION:

   (a) **FALSE** The error is related to the third order derivatives.

   (b) **FALSE** The traces for $z =$ constant are hyperbolas or two lines.

   (c) **TRUE** The $\mathbf{k}$-component of $\mathbf{N}$ vanishes for all $t \geq 0$ as shown below.

$$\mathbf{r}'(t) = \langle -\sin t, \cos t, 1\rangle \implies \|\mathbf{r}'(t)\| = \sqrt{2} \implies \mathbf{T}(t) = \frac{1}{\sqrt{2}}\langle -\sin t, \cos t, 1\rangle$$

$$\mathbf{T}'(t) = \frac{1}{\sqrt{2}}\langle -\cos t, -\sin t, 0\rangle \implies \|\mathbf{T}'(t)\| = \frac{1}{\sqrt{2}} \implies \mathbf{N}(t) = \langle -\cos t, -\sin t, 0\rangle$$

   (d) **FALSE** The planes do not intersect. Note that the normal vectors of the planes, $\left\langle \frac{1}{2}, 1, 4\right\rangle$ and $\langle 1, 2, 8\rangle$, are scalar multiples of one another and thus parallel implying that the planes to do not intersect.

   (e) **TRUE** If $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$, and the vectors are all nonzero, this means that $\mathbf{a}$ is orthogonal to the plane formed by $\mathbf{b}$ and $\mathbf{c}$, and thus must lie in the same plane as $\mathbf{b}$ and $\mathbf{c}$ since $\mathbf{b} \times \mathbf{c}$ is orthogonal to both $\mathbf{b}$ and $\mathbf{c}$ and to the plane in which they lie.

   (f) **TRUE** $r = \sqrt{x^2 + y^2}$ shows the equivalence of the Cartesian and cylindrical equations. Since $z = \rho\cos\phi$ and $r = \rho\sin\phi$, we have $\rho\cos\phi = -\rho\sin\phi \implies \tan\phi = -1 \implies \phi = 3\pi/4$ showing the equivalence of the cylindrical and spherical equations.

   (g) **FALSE** The magnitude of the normal component of the acceleration is proportional to both the curvature of the path and the square of the velocity, and since the curvature of a circle is the reciprocal of the radius we have $a_{\mathbf{N}} = \kappa\|\mathbf{v}\|^2 = \|\mathbf{v}\|^2/r$. The object on the circle of radius 4 will experience a smaller normal component of acceleration than the one on the circle of radius 1/4.

■

2. [2350/121222 (30 pts)] Let $\mathbf{V} = \left(x^2 + 4xy\right)\mathbf{i} - 6y\,\mathbf{j}$ be a velocity vector field.

   (a) [15 pts] Find the flow of $\mathbf{V}$ along the curve $\mathcal{C}$ given by $y = x^2 - x$ for $0 \leq x \leq 1$.

   (b) [15 pts] Find the total outward flux of $\mathbf{V}$, as a function of a and b, across the boundary of the rectangular region $0 \leq x \leq a,\ 0 \leq y \leq b$ where a, b are positive constants.

SOLUTION:

(a) A parameterization of the curve is $\mathbf{r}(t) = t\,\mathbf{i} + (t^2 - t)\,\mathbf{j},\ 0 \le t \le 1$. Then

$$\mathbf{V}[\mathbf{r}(t)] = \left[t^2 + 4t(t^2 - t)\right]\mathbf{i} - 6(t^2 - t)\,\mathbf{j} = \left(4t^3 - 3t^2\right)\mathbf{i} + \left(6t - 6t^2\right)\mathbf{j}$$

$$\mathbf{r}'(t) = \mathbf{i} + (2t - 1)\,\mathbf{j}$$

$$\mathbf{V}[\mathbf{r}(t)] \cdot \mathbf{r}'(t) = 4t^3 - 3t^2 + 12t^2 - 6t - 12t^3 + 6t^2 = -8t^3 + 15t^2 - 6t$$

$$\text{Flow} = \int_{\mathcal{C}} \mathbf{V} \cdot \mathrm{d}\mathbf{r} = \int_0^1 \left(-8t^3 + 15t^2 - 6t\right) \mathrm{d}t = \left(-2t^4 + 5t^3 - 3t^2\right)\Big|_0^1 = 0$$

(b) Use Green's Theorem. Let the rectangular region be denoted by $\mathcal{D}$ with boundary $\partial\mathcal{D}$.

$$\text{Flux} = \int_{\partial D} \left(x^2 + 4xy\right) \mathrm{d}y - (-6y)\,\mathrm{d}x = \iint_{\mathcal{D}} \left[\frac{\partial}{\partial x}\left(x^2 + 4xy\right) + \frac{\partial}{\partial y}(-6y)\right] \mathrm{d}A$$

$$= \int_0^a \int_0^b (2x + 4y - 6)\,\mathrm{d}y\,\mathrm{d}x$$

$$= \int_0^a \left(2xy + 2y^2 - 6y\right)\Big|_0^b \mathrm{d}x$$

$$= \int_0^a \left(2bx + 2b^2 - 6b\right) \mathrm{d}x$$

$$= \left(bx^2 + 2b^2x - 6bx\right)\Big|_0^a$$

$$= a^2b + 2ab^2 - 6ab$$

If doing this directly, note that there is no component of $\mathbf{V}$ normal to the x- or y-axis and thus no flux on those two sides of the rectangle. Only the sides $x = a$ ($\mathcal{C}_1$) and $y = b$ ($\mathcal{C}_2$) can contribute to the flux.

$$\mathcal{C}_1 : x = a, \mathrm{d}x = 0\,\mathrm{d}t;\ y = t, \mathrm{d}y = \mathrm{d}t \ \text{ with } 0 \le t \le b$$

$$\int_{\mathcal{C}_1} P\,\mathrm{d}y - Q\,\mathrm{d}x = \int_0^b \left(a^2 + 4at\right) \mathrm{d}t = a^2b + 2ab^2$$

$$\mathcal{C}_2 : x = a - t, \mathrm{d}x = -\,\mathrm{d}t;\ y = b, \mathrm{d}y = 0\,\mathrm{d}t \ \text{ with } 0 \le t \le a$$

$$\int_{\mathcal{C}_2} P\,\mathrm{d}y - Q\,\mathrm{d}x = \int_0^a -6b\,\mathrm{d}t = -6ab$$

$$\text{Flux} = \int_{\partial D} \mathbf{V} \cdot \mathbf{n}\,\mathrm{d}s = \int\limits_{\mathcal{C}_1 \cup \mathcal{C}_2} P\,\mathrm{d}y - Q\,\mathrm{d}x = a^2b + 2ab^2 - 6ab$$

■

3. [2350/121222 (20 pts)] Let $\mathbf{F} = (e^x \sin y - yz)\,\mathbf{i} + (e^x \cos y - xz)\,\mathbf{j} + (z - xy)\ \mathbf{k}$.

(a) [6 pts] Show that $\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$ for all closed paths in $\mathbb{R}^3$.

(b) [8 pts] Find the potential function $f(x, y, z)$ for the field $\mathbf{F}$.

(c) [6 pts] Calculate the work done by the force $\mathbf{F}$ moving an object along the line segment, $\mathcal{C}$, from $(0, \pi/2, -1)$ to $(1, \pi, 2)$.

SOLUTION:

(a) We can show this by showing that $\mathbf{F}$ is irrotational on a simply-connected domain. Since $\mathbf{F}$ is defined throughout $\mathbb{R}^3$, a simply-connected domain, we need to compute curl $\mathbf{F}$.

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^x \sin y - yz & e^x \cos y - xz & z - xy \end{vmatrix}$$

$$= [-x - (-x)]\,\mathbf{i} + [-y - (-y)]\,\mathbf{j} + [(e^x \cos y - z) - (e^x \cos y - z)]\,\mathbf{k}$$

$$= \mathbf{0}$$

(b)

$$\frac{\partial f}{\partial x} = e^x \sin y - yz \implies f(x,y,z) = \int (e^x \sin y - yz)\,\mathrm{d}x = e^x \sin y - xyz + g(y,z)$$

$$\frac{\partial f}{\partial y} = e^x \cos y - xz + g_y(y,z) = e^x \cos y - xz \implies g_y(y,z) = 0 \implies g(y,z) = h(z)$$

$$\implies f(x,y,z) = e^x \sin y - xyz + h(z)$$

$$\frac{\partial f}{\partial z} = -xy + h'(z) = z - xy \implies h(z) = \int z\,\mathrm{d}z = \frac{1}{2}z^2 + c$$

Thus the potential function is $f(x,y,z) = e^x \sin y - xyz + \frac{1}{2}z^2 + c$

(c) Use the Fundamental Theorem for Line Integrals.

$$\begin{aligned}
\text{Work} &= \int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{\mathcal{C}} \nabla f \cdot \mathrm{d}\mathbf{r} = f(1,\pi,2) - f(0,\pi/2,-1) \\
&= \left[e^1 \sin\pi - (1)(\pi)(2) + \frac{1}{2}(2)^2\right] - \left[e^0 \sin(\pi/2) - (0)(\pi/2)(-1) + \frac{1}{2}(-1)^2\right] \\
&= -2\pi + 2 - \left(1 + \frac{1}{2}\right) \\
&= -2\pi + \frac{1}{2}
\end{aligned}$$

■

4. [2350/121222 (22 pts)] Consider the vector field $\mathbf{F} = -3y\,\mathbf{i} + 3x\,\mathbf{j} + z^2\,\mathbf{k}$. Let $\mathcal{S}$ be that portion of the surface $3x^2 + 3y^2 - z^2 + 1 = 0$ lying above the plane $z = -\sqrt{13}$ and below the xy-plane.

   (a) [2 pts] Name the surface.

   (b) [10 pts] Compute the downward flux of the curl of $\mathbf{F}$ through $\mathcal{S}$ by directly evaluating $\iint_{\mathcal{S}} \boldsymbol{\nabla} \times \mathbf{F} \cdot \mathrm{d}\mathbf{S}$.

   (c) [10 pts] Compute the quantity in part (b) by evaluating an appropriate line integral.

SOLUTION:

(a) Hyperboloid of two sheets.

(b)

$$\boldsymbol{\nabla} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -3y & 3x & z^2 \end{vmatrix} = 6\,\mathbf{k}$$

We can consider $\mathcal{S}$ to be the level surface of $g(x,y,z) = 3x^2 + 3y^2 - z^2$ where $g = -1$. Then projecting the surface onto the xy-plane we have

$$\nabla g = \langle 6x, 6y, -2z \rangle$$

$$\mathbf{p} = \mathbf{k}$$

$$|\nabla g \cdot \mathbf{p}| = |-2z|$$

$$\text{region of integration } \mathcal{R} : 3x^2 + 3y^2 - \left(-\sqrt{13}\right)^2 = -1 \implies x^2 + y^2 \le 4$$

To properly orient the surface (downward pointing normal vector) we use $-\nabla g$ since $z < 0$ on $\mathcal{S}$, implying $\boldsymbol{\nabla} \times \mathbf{F} \cdot (-\nabla g) = 12z$. Also, $|\nabla \cdot \mathbf{p}| = -2z$, again since $z < 0$. Then

$$\iint_{\mathcal{S}} \boldsymbol{\nabla} \times \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iint_{\mathcal{R}} \frac{12z}{-2z}\,\mathrm{d}A = -6 \iint_{x^2+y^2 \le 4} \mathrm{d}A = -24\pi$$

(c) We use Stokes' Theorem and compute the flow of $\mathbf{F}$ along the boundary, ∂S, of $\mathcal{S}$, which is the circle of radius 2 in the plane $z = -\sqrt{13}$ centered on the z-axis. For proper orientation, ∂S must be traversed clockwise when looking downward. Thus,

$$\mathbf{r}(t) = \left\langle 2\sin t, 2\cos t, -\sqrt{13}\right\rangle,\ 0 \le t \le 2\pi$$

$$\mathbf{F}[\mathbf{r}(t)] = \langle -6\cos t, 6\sin t, 13\rangle$$

$$\mathbf{r}'(t) = \langle 2\cos t, -2\sin t, 0\rangle$$

$$\mathbf{F}[\mathbf{r}(t)]\cdot\mathbf{r}'(t) = -12\cos^2 t - 12\sin^2 t = -12$$

$$\iint_{\mathcal{S}} \boldsymbol{\nabla}\times\mathbf{F}\cdot \mathrm{d}\mathbf{S} = \int_{\partial\mathcal{S}} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_0^{2\pi} -12\,\mathrm{d}t = -24\pi$$

■

5. [2350/121222 (27 pts)] Let $\mathcal{S}$ be the portion of the surface $x^2 + y^2 + 2z = 16$ between the planes $z = 0$ and $z = 6$. Let $\mathbf{E} = \langle y, -x, 2z\rangle$ represent an electric field.

   (a) (2 pts) $\mathcal{S}$ is a portion of what quadric surface?

   (b) (10 pts) Compute the upward flux of the electric field through the surface by evaluating $\iint_{\mathcal{S}} \mathbf{E}\cdot\mathbf{n}\,\mathrm{d}S$ directly.

   (c) (15 pts) The surface $\mathcal{S}$ is not closed. However, Gauss' Divergence Theorem can still be applied to find the flux of $\mathbf{E}$ through $\mathcal{S}$. Apply the theorem to verify your answer to part (b).

SOLUTION:

(a) circular paraboloid

(b) We project the surface onto the xy-plane so that $\mathbf{p} = \mathbf{k}$. To find the region of integration, $\mathcal{R}$,

$$z = 0 : x^2 + y^2 = 16$$
$$z = 6 : x^2 + y^2 - 12 = 16 \implies x^2 + y^2 = 4$$

giving $\mathcal{R}$ as $4 \le x^2 + y^2 \le 16$. Furthermore,

$$g(x,y,z) = x^2 + y^2 + 2z \implies \nabla g = \langle 2x, 2y, 2\rangle \qquad \text{points upward for } \mathbf{n}$$

$$|\nabla g\cdot\mathbf{p}| = |2| = 2$$

$$\mathbf{E}\cdot\nabla g = \langle y, -x, 2z\rangle\cdot\langle 2x, 2y, 2\rangle = 2xy - 2xy + 4z = 4z$$

$$\iint_{\mathcal{S}} \mathbf{E}\cdot\mathbf{n}\,\mathrm{d}S = \iint_{\mathcal{R}} \frac{4z}{2}\,\mathrm{d}A = \iint_{\mathcal{R}} 2z\,\mathrm{d}A = \iint_{\mathcal{R}} \left(16 - x^2 - y^2\right)\mathrm{d}A \qquad \text{switch to polar coordinates}$$

$$= \int_0^{2\pi}\int_2^4 \left(16 - r^2\right) r\,\mathrm{d}r\,\mathrm{d}\theta = (2\pi)\left(8r^2 - \frac{r^4}{4}\right)\Bigg|_2^4 = 72\pi$$

(c) Let $\mathcal{S}_1$ be the disk $x^2 + y^2 \le 4$ at $z = 6$ and $\mathcal{S}_2$ be the disk $x^2 + y^2 \le 16$ at $z = 0$, then $\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}$ is the closed surface (boundary, $\partial\mathcal{E}$) of a solid region $\mathcal{E}$ to which we can apply Gauss' Divergence theorem. Thus

$$\iint\limits_{\mathcal{S}_1\cup\mathcal{S}_2\cup\mathcal{S}} \mathbf{E}\cdot \mathrm{d}\mathbf{S} = \iint_{\mathcal{S}_1} \mathbf{E}\cdot\mathbf{n}_1\,\mathrm{d}S + \iint_{\mathcal{S}_2} \mathbf{E}\cdot\mathbf{n}_2\,\mathrm{d}S + \iint_{\mathcal{S}} \mathbf{E}\cdot\mathbf{n}\,\mathrm{d}S = \iint_{\partial\mathcal{E}} \mathbf{E}\cdot \mathrm{d}\mathbf{S} = \iiint_{\mathcal{E}} \boldsymbol{\nabla}\cdot\mathbf{E}\,\mathrm{d}V$$

or

$$\iint_{\mathcal{S}} \mathbf{E}\cdot\mathbf{n}\,\mathrm{d}S = \iiint_{\mathcal{E}} \boldsymbol{\nabla}\cdot\mathbf{E} - \iint_{\mathcal{S}_1} \mathbf{E}\cdot\mathbf{n}_1\,\mathrm{d}S - \iint_{\mathcal{S}_2} \mathbf{E}\cdot\mathbf{n}_2\,\mathrm{d}S$$

The last integral on the right hand side is 0 since the vector field has no component normal to $\mathcal{S}_2$ $(z = 0)$. For $\mathcal{S}_1 (z = 6)$, we will project onto the xy-plane giving $\mathbf{p}_1 = \mathbf{k}$ and $\mathcal{R}_1$ as $x^2 + y^2 \le 4$. Also

$$g_1(x,y,z) = z \implies \nabla g_1 = \mathbf{k} \implies |\nabla g_1\cdot\mathbf{p}_1| = 1$$

with ∇g_1 providing the proper (outward pointing) orientation for $\mathcal{S}_1$. Then

$$\mathbf{E} \cdot \nabla g_1 = \langle y, -x, 2z \rangle \cdot \mathbf{k} = 2z$$

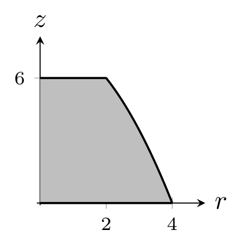
and

$$\iint_{\mathcal{S}_1} \mathbf{E} \cdot \mathbf{n}_1 \, \mathrm{d}S = \iint_{\mathcal{R}_1} 2z \, \mathrm{d}A = \iint_{\mathcal{R}_1} 2(6) \, \mathrm{d}A = (12)\text{area}(\mathcal{R}_1) = 12\pi(2^2) = 48\pi$$

The divergence of $\mathbf{E}$ is

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(-x) + \frac{\partial}{\partial z}(2z) = 2$$

Cylindrical coordinates will be best for the triple integral. The surface in these coordinates is $z = (16 - r^2)/2$. A sketch of a constant θ portion of the region $\mathcal{E}$ in the rz-plane follows.



$$\iiint_{\mathcal{E}} \boldsymbol{\nabla} \cdot \mathbf{E} \, \mathrm{d}V = 2 \iiint_{\mathcal{E}} \mathrm{d}V$$

$$= 2 \int_0^{2\pi} \int_0^6 \int_0^{\sqrt{16-2z}} r \, \mathrm{d}r \, \mathrm{d}z \, \mathrm{d}\theta = \int_0^{2\pi} \int_0^6 (16 - 2z) \, \mathrm{d}z \, \mathrm{d}\theta = 2\pi \left(16z - z^2\right) \Big|_0^6 = 120\pi$$

Finally then,

$$\iint_{\mathcal{S}} \mathbf{E} \cdot \mathbf{n} \, \mathrm{d}S = 120\pi - 48\pi = 72\pi$$

Note: Choosing a different order of integration in the triple integral results in

$$\iiint_{\mathcal{E}} \boldsymbol{\nabla} \cdot \mathbf{E} \, \mathrm{d}V = 2 \int_0^{2\pi} \int_0^2 \int_0^6 r \, \mathrm{d}z \, \mathrm{d}r \, \mathrm{d}\theta + 2 \int_0^{2\pi} \int_2^4 \int_0^{(16-r^2)/2} r \, \mathrm{d}z \, \mathrm{d}r \, \mathrm{d}\theta = 48\pi + 72\pi = 120\pi$$

■

6. [2350/121222 (30 pts)] A square metal plate occupies the region $0 \le x \le 20, 0 \le y \le 20$. The temperature of the plate is given by $T(x, y) = 3y^2 - 2y^3 - 3x^2 + 6xy$. In all of the parts below, be sure to use Calculus 3 concepts. No credit for reducing the problems to a single variable problem.

   (a) [8 pts] An ant is crawling around on the plate, following the curve $y = 1 + x^2$. At the point $(1, 2)$ in the direction of the ant's motion, how is the temperature changing with respect to

      i. distance?

      ii. time?

   (b) [10 pts] On this path, will the ant ever experience the warmest temperature in the plate's interior? Fully justify your answer.

   (c) [12 pts] Now suppose the ant is crawling along the portion of $x + y = 12$ lying on the plate. Find the highest and lowest temperatures the ant experiences on this new path.

SOLUTION:

(a) Parameterize the ant's path with $\mathbf{r}(t) = \langle t, 1 + t^2 \rangle \implies \mathbf{r}'(t) = \langle 1, 2t \rangle$. The path goes through $(1, 2)$ when $t = 1$. The ant's speed then, $\mathrm{d}s/\mathrm{d}t = \sqrt{5}$, so that a unit vector in the direction of the ant's path is $\frac{1}{\sqrt{5}}\langle 1, 2 \rangle$. The gradient of the temperature function is $\nabla T(x, y) = \left\langle -6x + 6y, 6y - 6y^2 + 6x \right\rangle$.

i.

$$\left.\frac{\mathrm{d}T}{\mathrm{d}s}\right|_{(1,2)} = \nabla T(1,2)\cdot\frac{1}{\sqrt{5}}\langle 1,2\rangle = \frac{1}{\sqrt{5}}\langle 6,-6\rangle\cdot\langle 1,2\rangle = -\frac{6}{\sqrt{5}}$$

ii.

$$\left.\frac{\mathrm{d}T}{\mathrm{d}t}\right|_{(1,2)} = \left.\frac{\mathrm{d}T}{\mathrm{d}s}\right|_{(1,2)}\left.\frac{\mathrm{d}s}{\mathrm{d}t}\right|_{(1,2)} = \left(-\frac{6}{\sqrt{5}}\right)\left(\sqrt{5}\right) = -6$$

(b) Need to find local maxima, if any exist, and decide if they are on the ant's path.

$$T_x = -6x + 6y \implies T_{xx} = -6$$

$$T_y = 6y - 6y^2 + 6x \implies T_{yy} = 6 - 12y$$

$$T_{xy} = 6$$

$$T_x = 0 \implies x = y \text{ so that } T_y = 0 = 12y - 6y^2 = 6y(2-y) \implies y = 0, 2$$

The critical points are thus $(0,0)$ and $(2,2)$. Using the second derivatives test

$$D(0,0) = (-6)(6-12(0)) - 36 = -72 < 0 \implies (0,0) \text{ is a saddle}$$

$$D(2,2) = (-6)(6-12(2)) - 36 = 72 > 0 \text{ and } T_{xx} < 0 \implies (2,2) \text{ is a local maximum}$$

The point $(2,2)$ is not on the ant's path so it will not experience it.

(c) Use Lagrange Multipliers on this closed bounded set. We have $g(x,y) = x + y \implies \nabla g = \langle 1,1\rangle$ leading to

$$\left.\begin{aligned} -6x + 6y &= \lambda \\ 6y - 6y^2 + 6x &= \lambda \end{aligned}\right\} \implies -6x + 6y = 6y - 6y^2 + 6x \implies x = \frac{1}{2}y^2$$

Using this in the constraint gives

$$\frac{1}{2}y^2 + y = 12$$

$$y^2 + 2y - 24 = (y-4)(y+6) = 0 \implies y = 4, -6 \implies x = 8 \text{ and } x = 18.$$

The only critical point on the plate is $(8,4)$. We also need to check the endpoints of the path, $(0,12)$ and $(12,0)$.

$$T(12,0) = -3(12^2) = -432$$

$$T(0,12) = 3(12)^2 - 2(12^3) = 12^2(3 - 2(12)) = 144(-21) = -3024$$

$$T(8,4) = 3(4)^2 - 2(4^3) - 3(8^2) + 6(8)(4) = 4^2\left[3 - 2(4)\right] - 8\left[3(8) - 6(4)\right] = -80$$

The warmest temperature the ant experiences is -80 and the coldest is -3024.

■